

Analysis Theme: Pure and Applied

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Analysis: from calculus to research

Analysis extends familiar ideas from calculus and linear algebra:

- Length, area and volume $\longrightarrow$ **measure and integration**.
- Vectors and matrices $\longrightarrow$ **function spaces and operators**.
- Rates of change $\longrightarrow$ **dynamical systems and PDEs**.

Why study it?

These tools connect pure mathematics with probability, physics, optimisation, biology and numerical analysis.

You do not need to be an analyst to use analysis!

Four modules: pure and applied

Semester I

- **Measure and Integration** (pure)
- **Dynamical Systems and Conservation Laws** (applied)

Semester II

- **Functional Analysis** (pure)
- **Elliptic and Parabolic PDEs** (applied)

Pure and applied analysis inform each other: new models lead to new mathematics, and abstract ideas help solve concrete problems.

Measure and Integration: a better integral

The Lebesgue integral gives a rigorous foundation for probability and a powerful way to integrate functions.

For $x \in [0, 1]$, consider

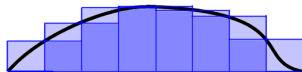
$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

- Every interval contains rational and irrational points: f is **not Riemann integrable**.
- The rationals have Lebesgue measure zero, so $\int_0^1 f(x) dx = 0$ in the Lebesgue sense.

Two ways to think about integration

Riemann integral:

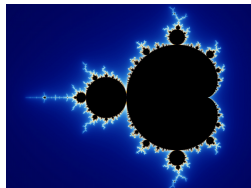
need to know measure of intervals / cubes $[a,b]^n$



Lebesgue integral:

need to know measure of
sublevel sets $\{x : f(x) < a\}$

sublevel sets can be wild:



Measure and Integration: the main ideas

- Constructing measures: from intervals to much more general sets.
- Lebesgue integration and convergence theorems: when can we interchange limits and integrals?
- Signed, product and Radon measures; Fubini's theorem for changing the order of integration.
- Fractal sets and Hausdorff dimension.

Where it leads

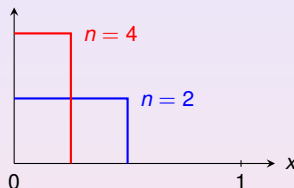
Probability, functional analysis and the study of irregular objects.

When can we interchange a limit and an integral?

On $[0, 1]$, let $f_n(x) = n \mathbf{1}_{(0, 1/n)}(x)$.

- At each fixed x , $f_n(x) \rightarrow 0$.
- But the area of every spike is one:

$$\int_0^1 f_n(x) dx = 1.$$



The missing ingredient

Pointwise convergence alone is not enough. The dominated convergence theorem supplies a sufficient condition: a common integrable bound.

D_SCL: how does a system evolve?

- **Dynamical systems:** equilibria, stability and bifurcations in ordinary differential equations.
- **Scalar conservation laws:**

$$u_t + (f(u))_x = 0.$$

- **Shock waves and entropy conditions:** what happens when smooth solutions break down, and how do we select a solution?

Ideas from ODEs also help us understand evolution equations in infinite-dimensional spaces.

A simple ODE system can have complex behaviour

The **Lorenz equations** arose from a simplified model of convection:

$$\dot{x} = \sigma(y - x),$$

$$\dot{y} = rx - y - xz,$$

$$\dot{z} = xy - bz.$$

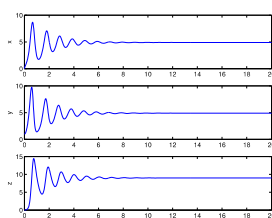
Here $\sigma, r, b > 0$ are parameters.

What changes as we vary a parameter?

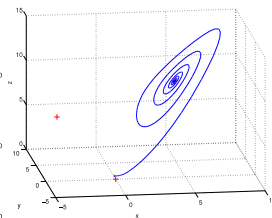
Keep $\sigma = 10$ and $b = 8/3$ fixed and compare trajectories for $r = 10$ and $r = 28$.

Lorenz system: approach to an equilibrium

Lorenz Eqs: $r = 10$, $\sigma = 10$, $b = 8/3$



$x(t), y(t), z(t)$

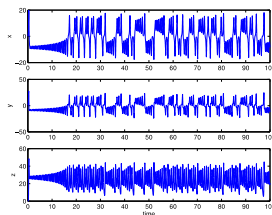


Phase space = $\mathbb{R}^3$

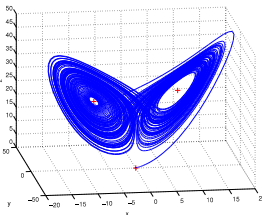
Solution converges to a 'fixed point'.

Lorenz system: chaotic behaviour

Lorenz Eqs: $r = 28$, $\sigma = 10$, $b = 8/3$



$x(t), y(t), z(t)$



Phase space = $\mathbb{R}^3$

Solution converges to classic 'chaotic attractor'.

Conservation laws: why do traffic jams move?

Let $\rho(x, t)$ be the density of cars on a road, with no entrances or exits.

Conservation of cars

$$\rho_t + (q(\rho))_x = 0, \quad q(\rho) = \rho v(\rho).$$

The flow q is density multiplied by speed.

- A simple model: $v(\rho) = v_{\max}(1 - \rho/\rho_{\max})$.
- Flow is small on an almost empty road *and* in a dense jam.
- Density waves can steepen into shocks: sharp transitions between free-flowing and congested traffic.

Cars move forwards, but the back of a growing jam can move backwards.

Functional Analysis: beyond finite dimensions

- **Banach and Hilbert spaces:** spaces of functions such as $C(X)$ and $L^p(X)$; the setting for Fourier analysis.
- **Linear operators and functionals:** the infinite-dimensional counterparts of matrices and linear forms.
- **Fundamental theorems:** Baire category, the open mapping theorem and the uniform boundedness principle.

The familiar language of linear algebra becomes a toolkit for analysis.

Functional Analysis: duality and compactness

- **Dual spaces:** study a space through its continuous linear functionals.
- **Weak and weak* convergence:** useful alternatives to convergence in norm.
- **Banach–Alaoglu:** the closed unit ball of a dual space is weak* compact.
- **Convexity:** the Krein–Milman theorem connects compact convex sets with their extreme points.

Why do we need weak convergence?

In the Hilbert space ℓ^2 of square-summable sequences, consider

$$e_1 = (1, 0, 0, \dots), \quad e_2 = (0, 1, 0, \dots), \quad \dots$$

- All have norm one, but $\|e_n - e_m\| = \sqrt{2}$ for $n \neq m$: there is **no norm-convergent subsequence**.
- Yet for every fixed $a \in \ell^2$,

$$|\langle e_n, a \rangle| = |a_n| \longrightarrow 0.$$

Thus e_n converges **weakly to zero**.

A useful change of viewpoint

Weaker notions of convergence help recover compactness when finite-dimensional intuition fails.

Functional Analysis: spectra and algebras

- Compact operators and their spectra.
- The spectral theorem for self-adjoint operators.
- Banach algebras and commutative C^* -algebras: Gelfand's theory.

Why it matters

Functional analysis is an essential tool in advanced PDE theory: it helps us find solutions and understand their properties.

Elliptic and Parabolic PDEs: two model equations

Elliptic: Poisson's equation

$$-\Delta u(x) = f(x).$$

Describes equilibrium: for example, a steady temperature distribution.

Parabolic: the heat equation

$$u_t(t, x) - \Delta u(t, x) = f(t, x).$$

Describes diffusion and evolution towards equilibrium.

Here $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$.

PDEs: the questions we ask

- **Existence:** is there a solution for the given data?
- **Uniqueness and stability:** is it the only one, and how does it depend on the data?
- **Qualitative behaviour:** bounds, regularity and long-time behaviour.

Maximum principles and energy estimates let us answer many of these questions without finding an explicit formula.

PDEs: the main tools

- Classical solutions of elliptic and parabolic equations; maximum principles, energy estimates and eigenfunctions.
- **Sobolev spaces**: functions with weak derivatives in L^p ; embedding and trace theorems.
- **Weak solutions and variational methods**: a bridge to functional analysis and the foundations of finite elements.

The emphasis is on the mathematical theory of PDEs.

The heat equation: an energy estimate

Suppose $u_t - \Delta u = 0$ in a bounded domain Ω , with $u = 0$ on the boundary. For a smooth solution, define

$$E(t) = \frac{1}{2} \int_{\Omega} |u(t, x)|^2 dx.$$

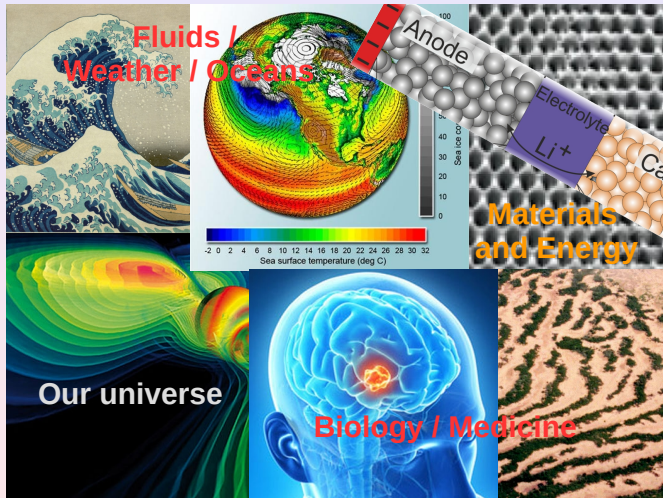
Multiply the equation by u and integrate by parts:

$$\frac{dE}{dt} = - \int_{\Omega} |\nabla u|^2 dx \leq 0.$$

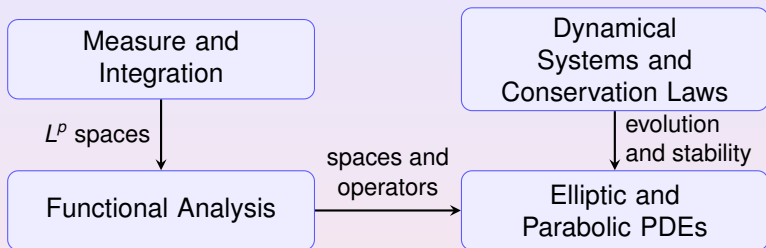
- The squared L^2 norm decreases: diffusion dissipates energy.
- Apply the same estimate to the difference of two solutions with the same data: they must agree.

One estimate gives information without an explicit solution.

Some applications of PDEs



How the four modules fit together



The energy estimate brings these ideas together

Integration measures the size of a solution; function spaces express the bounds; the PDE describes how the solution evolves.

Prerequisites for the pure modules

Undergraduate real analysis

Sequences and series; pointwise and uniform convergence; continuity and differentiability; open, closed and compact sets; countability.

For Functional Analysis

Measure theory and some Fourier analysis help with the examples. You can still benefit from the course without having studied measure theory, though some examples will be less familiar.

Prerequisites for the applied modules

- Undergraduate ordinary differential equations.
- Single- and multivariable calculus and real analysis.
- Linear algebra.

Elliptic and Parabolic PDEs is largely self-contained; DSCL provides useful motivation and background.

Choosing your modules

Think about the tools your research will need, and discuss your choices with your supervisor and the teaching teams.

Learning and assessment

During the course

Work through the lecture exercises and use tutorials to discuss them.
Read the available notes in advance and bring questions to lectures.

- **Measure and Integration:** two equally weighted assignments, around three questions each.
- **Functional Analysis:** two equally weighted assignments.
- **Elliptic and Parabolic PDEs:** two equally weighted assignments.
- **DSCL:** two equally weighted assignments.

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