

QFT for mathematicians

- semester 2 Tuesdays 9-11
- lecturers : Lotte Hollands, Yegor Zenkevich, Loïc Bramley
- core course on AGQ programme /
SMSTC supplementary course
- prerequisites: differential + algebraic topology
(some physics background convenient but not required)

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
& + i\bar{\psi}\not{D}\psi + \text{h.c.} \\
& + \chi_i Y_{ij} \chi_j \phi + \text{h.c.} \\
& + |D_\mu \phi|^2 - V(\phi)
\end{aligned}$$

*Standard Model
Lagrangian*

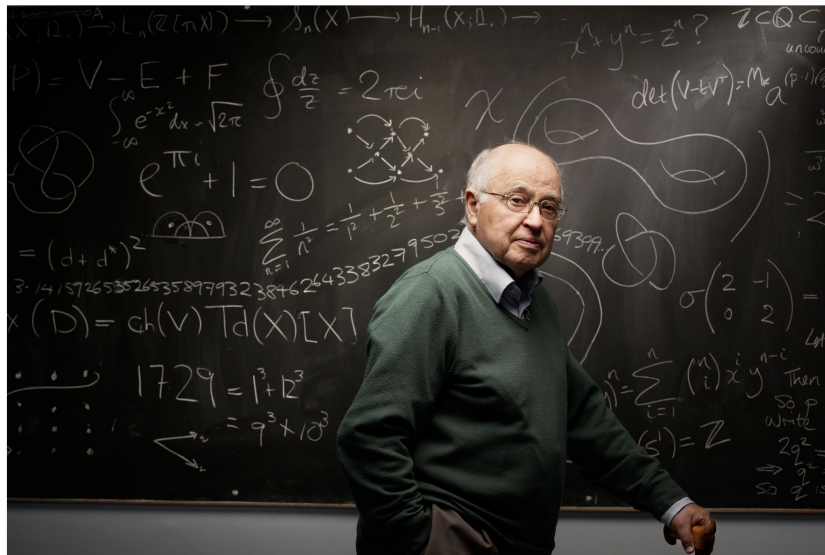
Gauge Fields enormously important in fundamental physics
particle physics, general relativity, ...

But also in mathematics:

Donaldson-Witten invariants of 4-manifolds

Knot invariants and Chern-Simons theory

Hitchin integrable system and ADHM moduli space

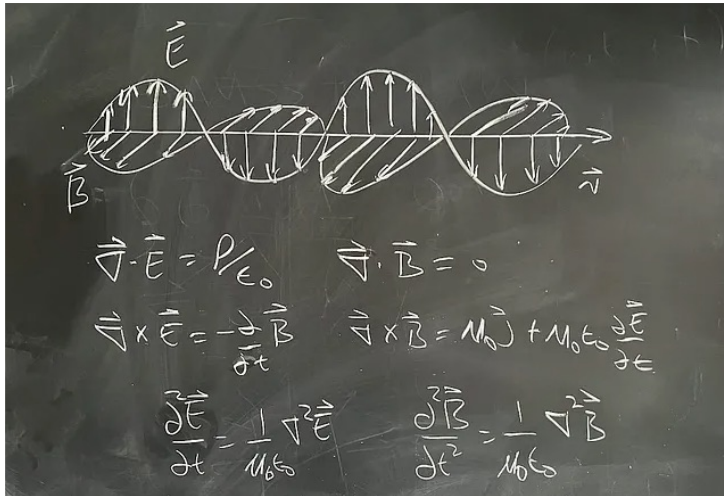
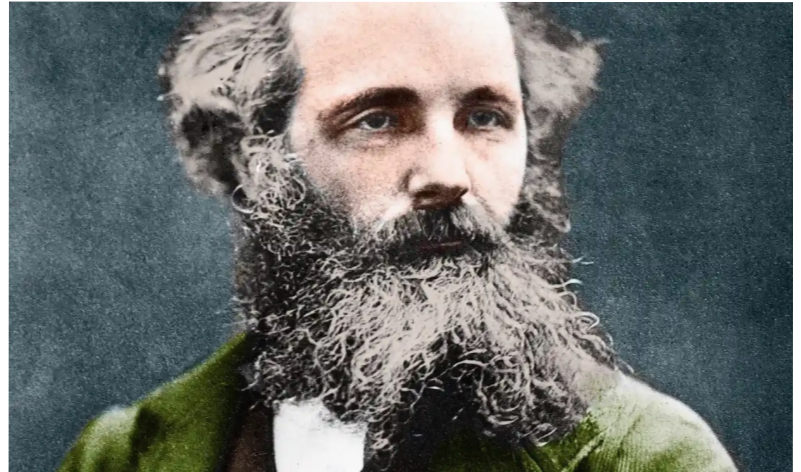


Michael Atiyah

(1929 - 2019)

Some history:

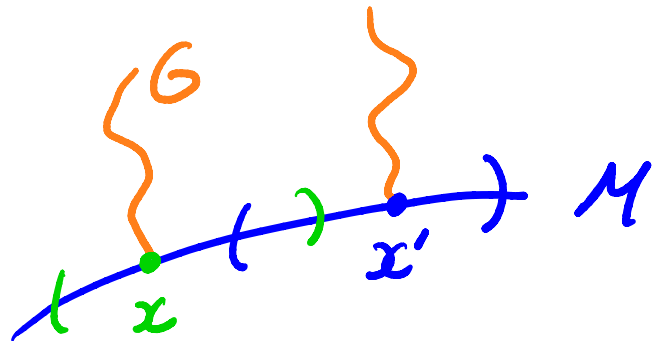
James Clark Maxwell
(1831-1879)



$\leadsto$ $U(1)$ gauge theory

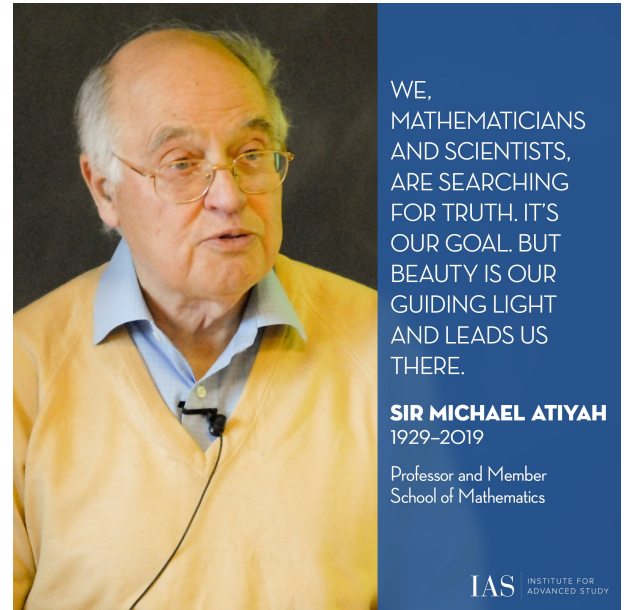
Idea gauge theory:

- there is a redundancy in the formalism: the formalism depends on "gauges", but reality does not
- gauge freedom is often captured by compact Lie groups, such as $U(1) \times SU(2) \times SU(3)$
- mathematically, this is described in terms of principal bundles:



Michael Atiyah (1929-2019)

*Mathematician who helped shape
the modern viewpoint that
geometry and topology are
fundamental to understanding
quantum field theory*



*"Interaction mathematics - physics is the most
productive area in science today"*

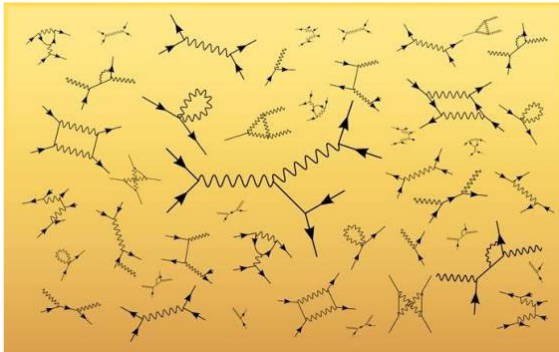
Course content:

theme of each lecture is an important concept in QFT

and related mathematical developments:

1. perturbative QFT: path integral + Feynmann diagrams

→ formalised by Costello- Gwilliam as deformation
quantisation of factorisation algebra



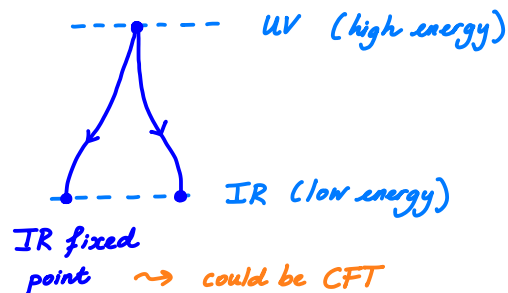
→ Application: 3d Chern-Simons theory
and knot invariants / 3-mfd invariants

$$\int_M dA \wedge A + \frac{2}{3} A \wedge A \wedge A$$

Hilbert space = quantisation of moduli space of flat
connections on Riemann surface

2. renormalisation + RG flow (universality)

→ coupling constants $g(E)$ depend on energy E

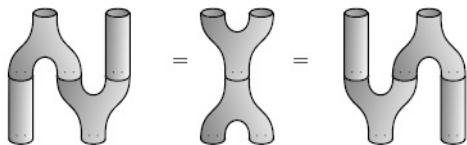


→ Application: Donaldson invariants = Seiberg-Witten invariants

↳ invariants on moduli spaces of anti-selfdual connections on 4-manifolds $\sim$ topologically twisted Seiberg-Witten theory

3. Locality

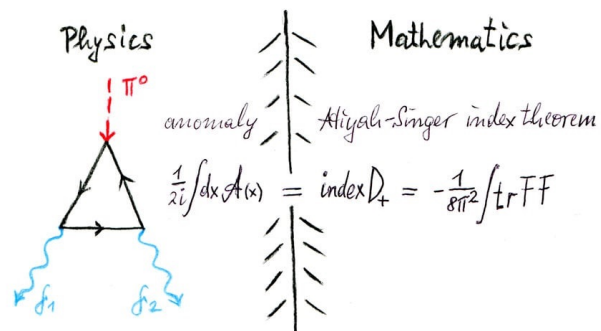
→ QFT's are determined by what they assign to arbitrary small pieces of space-time, together with gluing rules



→ Application: extended TFT's

e.g. 2d TFT is functor from 2d cobordisms to commutative Frobenius algebras

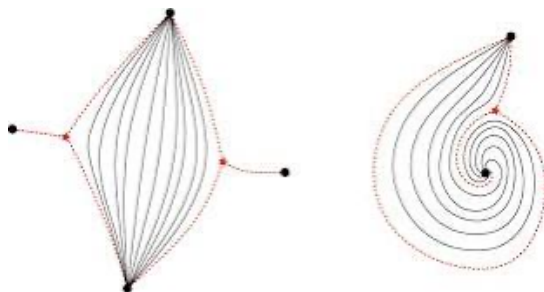
4. Symmetries & anomalies $\leadsto$ Application: Atiyah-Singer index thm



5. Twists and localisation

$\leadsto$ Application: Computation of Donaldson invariants
geometric Langlands
BV formalism and derived geometry

6. BPS states, compactification and dualities



Application: Donaldson-Thomas invariants,
wall-crossing,
homological mirror symmetry