

SMSTC: Probability and Statistics

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- Probability and Statistics
- Course outlines and teaching teams
- Prerequisites
- Assessment
- Feedback

“.. the true logic for this world is the calculus of Probabilities, which takes account of the magnitude of the probability which is, or ought to be, in a reasonable man's mind.”

– James Clerk Maxwell (1850)

From the book “Probability theory: the logic of science” by E.T.Jaynes

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Statistics may be defined as “a body of methods for making wise decisions in the face of uncertainty.”

– W.A. Wallis

- **mathematical modelling of uncertainty**: random events and random processes evolving in time
- crucial to understand **dependence** between different elements of the model
- in practice, driven by understanding **properties of experimental observations**
- correct **measure of uncertainty for the decision making** process.

Statistical inference $\approx$ Probability⁻¹?

- **Probability theory:** for a specific fully-described probability distribution, we care about data generation and its properties.

Example (Probability problem)

Let $\mathbf{Y} = (Y_1, \dots, Y_{10}) \stackrel{i.i.d.}{\sim} \mathcal{N}(4.5, 0.1)$ be r.v.'s. For instance, we could ask about

- $\mathbb{P}[Y_3 > 1.5]$,
- distribution of the average $\bar{Y} = \frac{1}{10} \sum_{i=1}^{10} Y_i$.

Probability and Statistics

- **Statistical inference:** for a fixed and observed data vector $\mathbf{y}$, we care about guessing properties of the distribution that generated the data. e.g., unknown parameters that help us make decisions.

Example (Statistical-inference problem, e.g., Google Maps rating)

Let $\mathbf{Y} = (Y_1, \dots, Y_{10}) \stackrel{i.i.d.}{\sim} \mathcal{N}(\theta, 0.1)$ be r.v.'s with unknown mean parameter θ . We observe a data vector, $\mathbf{y}$ which is a realisation of $\mathbf{Y}$, say with $n = 10$ data points $y_1 = 4.55, y_2 = 4.26, \dots, y_{10} = 4.03$. The empirical mean is $\bar{y} = 4.1$. Some questions arise:

- Can we give a good estimate $\hat{\theta}$ of θ ?
- Can we give a measure of (un)certainty about our guess?
- Is $\hat{\theta} = 4.5$ a good/reasonable estimate?
- Would $\hat{\theta} = 4.5$ still be a good estimate if $\bar{y} = 4.1$ but $n = 2000$?

Statistical inference is what your brain does. **All the time.**

Modules:

- Foundations of Probability (Semester 1)
- Stochastic Processes (Semester 2)

Foundations of Probability (Semester 1)

- **Fundamentals:** probability spaces, σ -algebras, probability measures, conditioning and independence
- **Random variables** and their distributions, important special distributions (binomial, Poisson, geometric, normal, exponential etc.)
- **Convergence** and **limit theorems**
- **Conditional expectation** and **martingales**
- **Renewal theory**

Stochastic Processes (Semester 2)

- **Markov chains** and **processes**, **Poisson processes**
- **Applications**, including connections to statistics and graph theory
- **Brownian motion** and **stochastic calculus**

Stochastic Processes (Semester 2)

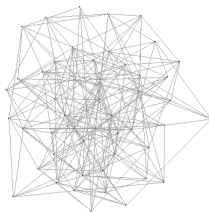
Suppose we have n vertices/nodes.

Each pair of vertices is joined by an edge/link with probability p , independently of all other pairs of vertices.

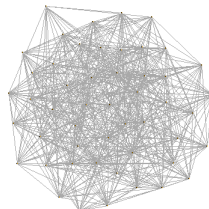
This is the Erdős–Rényi random graph $G(n, p)$. It can be used to model a ‘typical’ (or ‘unstructured’ or ‘random’) communication (or power, or distribution, or biological, or ...) network, for example.



$p = 0.05$



$p = 0.2$



$p = 0.5$

Let $p = c/n$. Then (under some mild conditions on c) $G(n, p)$ contains a path of length at least *constant* $\times n$ with probability close to 1, for large enough n .

This is proved by analysing an algorithm which explicitly constructs such a path, and exploiting the *Markovian* structure present in the algorithm.

Stochastic Processes (Semester 2)

Let K_n be the complete graph, with n vertices and an edge between each pair of vertices. Suppose we colour each edge of K_n either red or blue.

There is a colouring of K_n which contains at most $\binom{n}{a} 2^{1-\binom{a}{2}}$ monochromatic copies of the complete graph K_a .

Stochastic Processes (Semester 2)

Let K_n be the complete graph, with n vertices and an edge between each pair of vertices. Suppose we colour each edge of K_n either red or blue.

There is a colouring of K_n which contains at most $\binom{n}{a} 2^{1-\binom{a}{2}}$ monochromatic copies of the complete graph K_a .

One can prove this by

- Randomly colouring K_n (each edge is red with probability $1/2$, or blue otherwise, independently of the other edges);
- Calculating that the average number of monochromatic copies of K_a is $\binom{n}{a} 2^{1-\binom{a}{2}}$; and
- Concluding that there must exist a colouring with at most this many monochromatic copies of K_a .

This year:

- Foundations of Probability: Sergey Foss (Heriot–Watt)
- Stochastic Processes: Burak Buke (Edinburgh)

For questions related to the Probability modules, feel free to get in touch with Sergey (S.Foss@hw.ac.uk) or Burak (B.Buke@ed.ac.uk).

Tutorial support will be arranged locally.

Probability: Prerequisites

- Elements of mathematical analysis, linear algebra and combinatorics at undergraduate level.
- For Stochastic Processes, in addition: Probability theory, either at undergraduate level or from Foundations of Probability.

Probability: Assessment

Each module is assessed by two written assignments.

Approximate deadlines:

- Foundations of Probability: mid-November and early January.
- Stochastic Processes: mid-February and end of March.

Assignments will be available at least two weeks before the deadline.

Solutions for (at least) one assignment from each module should be prepared using $\text{\LaTeX}$.

Aims:

- Model fitting from experimental data: How do we select an appropriate model? How do we fit parameters to a given data set? How do we handle imperfect (missing/contaminated/...) data? How do we quantify uncertainty in our estimates?
- Testing plausibility of given conjectures.
- Simulation of intractable probability distributions.

Modules:

- Regression and Simulation Methods (Semester 1)
- Modern Regression and Bayesian Methods (Semester 2)

Regression and Simulation Methods (Semester 1)

Linear model:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \epsilon_i,$$

for $i = 1, \dots, n$ (where n is the sample size), and where $\epsilon_1, \dots, \epsilon_n$ are independent and identically distributed with $\epsilon_1 \sim N(0, \sigma^2)$.

More succinctly

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}).$$

Residual Sum of Squares:

$$\text{RSS} = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}),$$

minimised by choosing

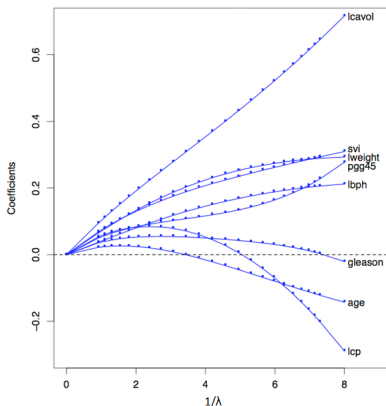
$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}.$$

Regression and Simulation Methods (Semester 1)

What happens when $\mathbf{X}^T \mathbf{X}$ is singular?

One possible solution: Ridge regression

$$\hat{\beta}^{\text{ridge}} = (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1} \mathbf{X}^T \mathbf{y}.$$



Regression and Simulation Methods (Semester 1)

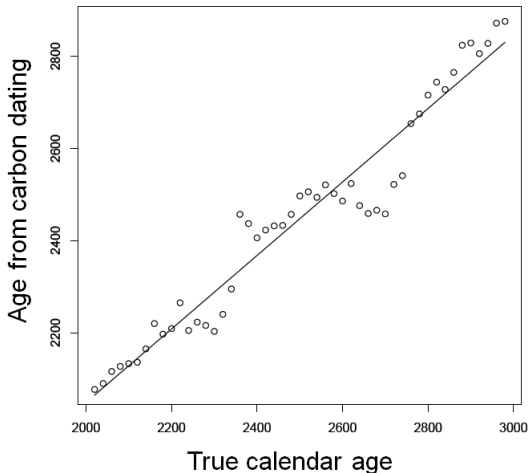
- **Introduction to R**
- **Linear models:** Estimation, testing, model checking, factors, model fitting in R. Analysis of simple designed experiments. Case studies.
- **Likelihood and optimisation:** Likelihood principles and key distributional results. Examples. Newton's method for optimisation. Two-parameter likelihoods. General optimisation methods. Implementation in R.
- **Generalised linear models:** Exponential family. Link functions. Examples. Iteratively weighted least squares. Model fitting in R. Case studies.
- **Simulation and bootstrapping:** Non-parametric bootstrap; confidence intervals; implementation in R. Parametric bootstrap. Simulation methods and implementation in R.
- **Case study**

Regression and Simulation Methods (Semester 1)

- The first half of Regression and Simulation Methods will be run as an online audio/video course. It covers what for many will be revision, and this flexible form of delivery allows participants to study different parts of the material at a speed and depth appropriate for them.
 - We ask you to check the course materials on the SMSTC website. If any of it is unfamiliar, you can view the relevant lectures, and attempt the related tutorial questions.
- Regular Zoom sessions will begin in the sixth session.

Modern Regression and Bayesian Methods (Semester 2)

Radiocarbon data: high precision measurements of Carbon-14 in Irish oak, used to construct a calibration curve (here with line of best fit)

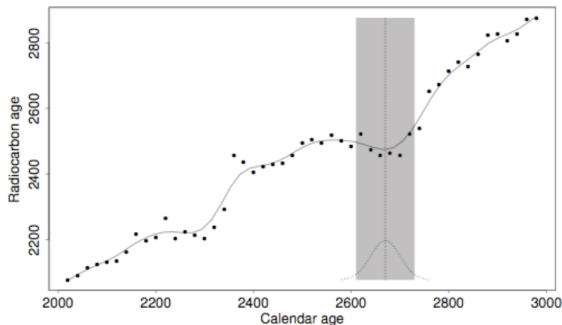


Modern Regression and Bayesian Methods (Semester 2)

One solution to non-linearity: *local* linear regression. Solve

$$\min_{\alpha, \beta} \sum_{i=1}^n \{y_i - \alpha - \beta(x_i - x)\}^2 w(x_i - x; h),$$

for a weight function w , and take $\hat{\alpha}$ as the estimate at x .



Parameter estimation framework

Let $\mathbf{Y} = (Y_1, \dots, Y_n) \sim p(\cdot | \theta)$ – likelihood, for some $\theta \in \Theta \subseteq \mathbb{R}^p$.

Bayesian model and Bayes estimator of θ :

Prior distribution: $\theta \sim p(\theta)$, $\theta \in \Theta$ – density of the prior distribution.

Posterior distribution: $p(\theta | \mathbf{Y}) \propto p(\mathbf{Y} | \theta) p(\theta)$, $\theta \in \Theta$.

Bayes estimator : for a given a loss function $Q(\hat{\theta}, \theta)$,

$$\hat{\theta} = \arg \min_{\hat{\theta} \in \Theta} \int Q(\hat{\theta}, \theta) p(\theta | \mathbf{y}) d\theta.$$

E.g.

- $Q(x, y) = (x - y)^2$ - posterior mean $\hat{\theta} = E(\theta | \mathbf{y})$
- $Q(x, y) = |x - y|$ - posterior median
- $Q(x, y) = I(x = y)$ - maximum a posteriori estimator (MAP)

For an appropriate loss function, it can be a decision, credible interval/region ...

Modern Regression and Bayesian Methods (Semester 2)

- **Random effects models:** Methods for linear and non-linear mixed effects models. Case studies.
- **Modern regression:** Density estimation. Non-parametric regression. Bandwidth selection. Examples. Additive models. The backfitting algorithm. Examples.
- **Bayesian methods:** Priors and posteriors. Prior sensitivity. Marginal distributions.
- **Markov chain Monte Carlo:** Metropolis-Hastings algorithm. Gibbs sampler. Convergence, burn-in, mixing properties, tuning parameters. WinBUGS. MCMC simulations in R. Examples. Advanced topics: eg, random effects, missing data, model selection.
- **Case study**

Semester 1 (Regression and Simulation Methods):

- Oliver Stoner (Glasgow)
- Claudia Colarin (Edinburgh)

Semester 2 (Modern Regression and Bayesian Methods):

- Oliver Stoner (Glasgow)
- Fergus Chadwick (St Andrews)

For questions related to the Statistics modules, feel free to get in touch with Oliver (Oliver.Stoner@glasgow.ac.uk), Claudia (claudia.collarin@ed.ac.uk) or Fergus (fergusjchadwick@st-andrews.ac.uk).

Tutorial support will be arranged locally.

Statistics: Prerequisites

- Basic concepts in probability (elementary probability distributions), statistics (idea of estimation, confidence intervals, hypothesis tests), calculus, and linear algebra. These would usually be provided in first undergraduate courses.
- For Modern Regression and Bayesian Methods: the semester 1 course (Regression and Simulation Methods), or equivalent.

Regression and Simulation Methods:

- One written assignment (based on the final five lectures), deadline in early-mid January. The assignment will be available by mid-December.

Modern Regression and Bayesian Methods:

- Two written assignments, one after each block of five lectures (around mid February and end of March). Assignments will be available at least two weeks before the deadline.

- If you have any questions/concerns, get in touch with us or another member of the teaching team.
- Feedback and questions are encouraged during lectures.
- Please don't wait for the end of the module!