

SMSTC Supplementary module

An introduction to Hopf algebras over fields

Andrew Baker (University of Glasgow)

last updated 26/09/2026

Outline

Prerequisites: As background, knowledge of the following would be useful: basic linear algebra and tensor products, rings, modules and representation theory of groups, as well some familiarity with category theory, commutative diagrams and homological algebra.

Learning outcomes: As well as exposure to important topics in modern abstract algebra, this is a good place to gain familiarity with the use of category theory in mathematics.

Who might be interested: Mathematicians and theoretical physicists working on representation theory of finite dimensional algebras and finite groups, knot theory, Lie theory, algebraic topology, algebraic geometry, non-commutative geometry, etc.

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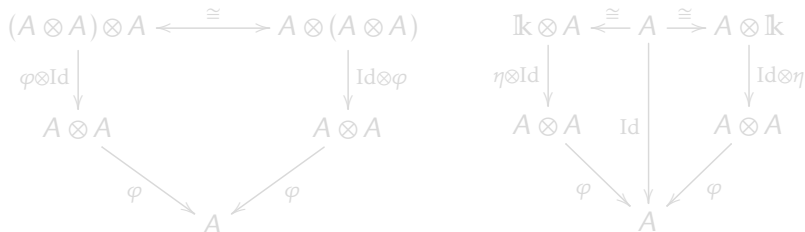
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Practicalities: There will be opportunities for students to give talks on topics that particularly interest them and those requiring credit will be expected to give the equivalent of a one hour talk (as one talk or two shorter talks).

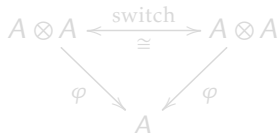
Course material: I have written a set of notes which contains a substantial list of references including many text books some of which are available from university libraries in pdf form. Participants are encouraged to read around the subject and in particular use exercise sets in books to gain better understanding of the material. Although there will be no formal exercise sets, if necessary we could discuss problems in informal tutorials as required.

Quick introduction

A $\mathbb{k}$ -algebra (A, φ, η) over a field $\mathbb{k}$ is a *monoid* in the monoidal category $(\mathbf{Vect}_{\mathbb{k}}, \otimes)$, i.e., a $\mathbb{k}$ -vector space A with a *product* $\varphi: A \otimes A \rightarrow A$ and *unit* $\eta: \mathbb{k} \rightarrow A$, which make the following diagrams in $\mathbf{Vect}_{\mathbb{k}}$ commute.



A is *commutative* if also the following diagram commutes.



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$$\begin{array}{ccc}
 (A \otimes A) \otimes A & \xleftarrow{\cong} & A \otimes (A \otimes A) \\
 \varphi \otimes \text{Id} \downarrow & & \downarrow \text{Id} \otimes \varphi \\
 A \otimes A & & A \otimes A \\
 \searrow \varphi & & \swarrow \varphi \\
 & A &
 \end{array}$$

$$\begin{array}{ccccc}
 \mathbb{k} \otimes A & \xleftarrow{\cong} & A & \xrightarrow{\cong} & A \otimes \mathbb{k} \\
 \eta \otimes \text{Id} \downarrow & & \downarrow \text{Id} & & \downarrow \text{Id} \otimes \eta \\
 A \otimes A & & A & & A \otimes A \\
 \searrow \varphi & & & & \swarrow \varphi \\
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A is *commutative* if also the following diagram commutes.

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 A \otimes A & \xleftarrow[\cong]{\text{switch}} & A \otimes A \\
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The dual notion is that of a $\mathbb{k}$ -coalgebra, which is a triple (C, ψ, ε) , with C a $\mathbb{k}$ -vector space, a coproduct $\psi: C \rightarrow C \otimes C$, and a counit $\varepsilon: C \rightarrow \mathbb{k}$ fitting into the commutative diagrams shown.

$$\begin{array}{ccc}
 (C \otimes C) \otimes C & \xleftarrow{\cong} & C \otimes (C \otimes C) \\
 \uparrow \psi \otimes \text{Id} & & \uparrow \text{Id} \otimes \psi \\
 C \otimes C & & C \otimes C \\
 \swarrow \psi & & \searrow \psi \\
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 C \otimes C & & C & & C \otimes C \\
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 & C & & &
 \end{array}$$

This says that (C, ψ, ε) is a comonoid in $\mathbf{Vect}_{\mathbb{k}}$.

C is cocommutative if also the following diagram commutes.

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This says that (C, ψ, ε) is a *comonoid* in $\mathbf{Vect}_{\mathbb{k}}$.

C is *cocommutative* if also the following diagram commutes.

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A $\mathbb{k}$ -Hopf algebra H is a $\mathbb{k}$ -vector space which is both an algebra and a coalgebra together with an antipode $\chi: H \rightarrow H$, so that all of this structure interacts in a certain way. A Hopf algebra is a *group object* in the category of coalgebras or a *cogroup object* in the category of algebras, so Hopf algebras generalise groups!

Some examples:

- ▶ The *group algebra* $\mathbb{k}[G]$ of a group G is a cocommutative Hopf algebra with basis G , product $\varphi(g' \otimes g'') = g'g''$, coproduct $\psi(g) = g \otimes g$, antipode $\chi(g) = g^{-1}$.
- ▶ If G is a topological group (e.g., a Lie group), $H_*(G; \mathbb{k})$ and $H^*(G; \mathbb{k})$ are Hopf algebras.
- ▶ If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $U(\mathfrak{g})$ is a cocommutative Hopf algebra.
- ▶ Coordinate algebras of affine group schemes are commutative Hopf algebras.
- ▶ Examples from combinatorics.
- ▶ Quantum groups are Hopf algebras which are neither commutative nor cocommutative.

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Outline of course content

Depending on time and the audience's interests, I expect to discuss most of the topics below.

- ▶ Some category theory: adjoint functors, monoidal categories (vector spaces over a field as an important example).
- ▶ Algebras and coalgebras over a field; bialgebras and Hopf algebras.
- ▶ Lots of examples.
- ▶ SubHopf algebras, adjoint actions and normal subHopf algebras.
- ▶ Modules and comodules, representation theory of a Hopf algebra.
- ▶ Hopf modules and finite dimensional Hopf algebras; every finite dimensional Hopf algebra is a Frobenius algebra.
- ▶ Quantum Groups.
- ▶ If time permits: other topics such as homological algebra for modules over Hopf algebras.