

Introduction to "Conformal Field Theory and Vertex Operator Algebras"

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Conformal Field Theory = Quantum Field Theory invariant under
conformal transformations

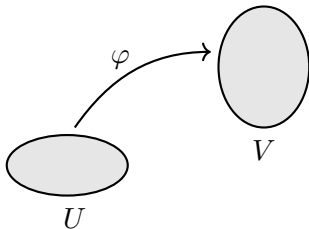
Conformal transformations

Consider a plane with a Riemannian metric

$$g = g_{\mu\nu} dx^\mu dx^\nu$$

A conformal map of a domain $U \subset \mathbb{R}^2$ is a diffeomorphism

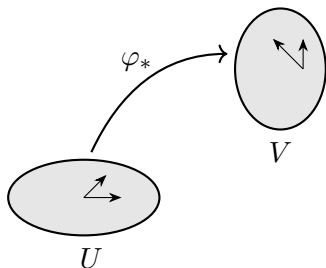
$$\varphi : U \rightarrow V$$



for which

$$\varphi^* g = \Lambda g \quad \Lambda = \Lambda(x) > 0$$

Conformal transformations in general do not preserve distances between points but preserve angles between vectors in tangent space



The group of conformal transformations

Among conformal transformations there is a subgroup of rigid motions

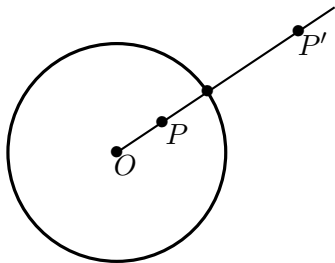
$$z \mapsto e^{i\alpha} z + B$$

and an even larger subgroup of affine transformations

$$z \mapsto Az + B, \quad A, B \in \mathbb{C}, \quad A \neq 0$$

Inversions

$$\mathcal{I} : z \mapsto \frac{R^2}{\bar{z}}$$



Inversions are conformal transformations defined on $\mathbb{C} \setminus \{\text{pt}\}$. A holomorphic version

$$C \circ \mathcal{I} : z \rightarrow \frac{1}{z}, \quad R = 1.$$

Möbius transformations

Affine transformations plus the holomorphic inversion generate the group of Möbius transformations:

$$z \mapsto \frac{az + b}{cz + d}.$$

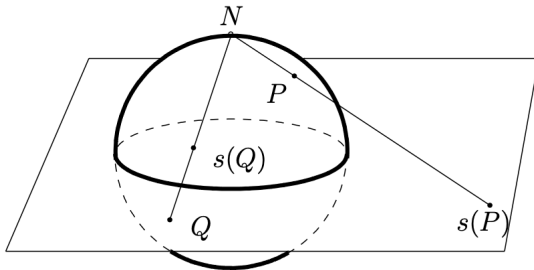
As a group it is isomorphic to a quotient group

$$\mathrm{PSL}(2, \mathbb{C}) := \mathrm{SL}(2, \mathbb{C}) / \{\pm 1\} \quad (1)$$

where $\mathrm{SL}(2, \mathbb{C})$ is the group of complex 2×2 matrices with determinant 1.

Riemann sphere

While Möbius transformations are in general defined on $\mathbb{C} \setminus \{\text{pt}\}$ they are extended to conformal bijections of the **Riemann sphere** that is a compactification of the complex plane.



A quantum field theory is a collection of functions

$$\langle \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) \dots \mathcal{O}_n(z_n) \rangle : \mathbb{C}^n \rightarrow \mathbb{C}$$

that satisfy certain axioms. For example these functions are real-analytic and have singularities only when some of the arguments coincide.

Covariance under Möbius transformations

In conformal field theories the correlation functions transform covariantly under conformal transformations

$$z \mapsto \varphi(z) = \frac{az + b}{cz + d}$$

$$\begin{aligned} & \langle \mathcal{O}_1(\varphi(z_1)) \dots \mathcal{O}_n(\varphi(z_n)) \rangle \\ &= |cz_1 + d|^{\Delta_1} \dots |cz_n + d|^{\Delta_n} \langle \mathcal{O}_1(z_1) \dots \mathcal{O}_n(z_n) \rangle \end{aligned}$$

These equations have some resemblance to the transformation property of modular forms.

More structure comes from infinitesimal conformal transformations. Those are generated by vector fields

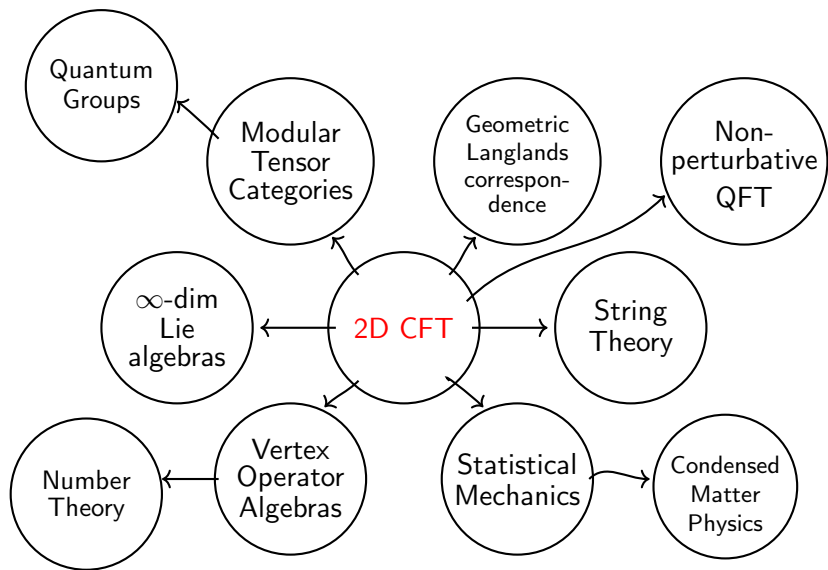
$$l_n = -z^n \partial_z - \bar{z}^n \partial_{\bar{z}} \quad n \in \mathbb{Z}, \quad z \neq 0$$

They satisfy the commutation relations

$$[l_n, l_m] = (n - m)l_{n+m}, \quad [\bar{l}_n, \bar{l}_m] = (n - m)\bar{l}_{n+m}$$

Each pair is called **Witt algebra** and is a complexification of the Lie algebra of the group of diffeomorphisms of a circle. In QFT one obtains a representation of its central extension called **Virasoro algebra**

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{\delta_{n+m,0}}{12}(n^3 - n)\mathcal{C}, \quad [\mathcal{C}, L_n] = 0$$



Mathematics prerequisites for the course:

- Basic group theory
- Basic complex analysis (calculating contour integrals via residues)
- Basic functional analysis (Hilbert space, inner product, Riesz representation theorem, bounded, Hermitian linear operators)
- Basic differential geometry (tangent space, forms and vector fields, metric, push-forward and pull-back maps)
- Analysis (a notion of distribution, distributional derivative, Dirac's delta function)

Physics prerequisites for the course:

- Some acquaintance with Quantum Mechanics is expected (Hilbert space, bra- and ket-vector notation, Hamiltonian, Schroedinger equation)
- No prior knowledge of quantum field theory is assumed although some acquaintance would be helpful. The exposition is axiomatic in spirit but not overly formalised. Several general concepts and constructions will be explained: correlation functions, Ward identities, canonical quantisation, GNS construction of Hilbert space from correlation functions.

Course organisation

The course starts with 6 introductory lectures on 2D CFT (delivered by AK) followed by 2 lectures introducing VOA delivered by Ioana Coman-Lohi



Reading projects on more advanced topics such as conformal blocks and AGT conjecture, modular invariance and rational VOA, Wess-Zumino-Witten theories and Kac-Moody algebras.

Assessment: based on a short presentation.

Course Notes: There are detailed course notes to be used together with the lectures. They are meant to complement the lectures with many additional details to be found in the notes. I suggest reading the material in the notes before and after the lectures. Also some additional reading is suggested in the notes.

Exercises: 9 tutorial sheets (with model solutions) are offered for self study. While the lectures will focus on conceptual issues and won't have many calculations done in detail the problems in the sheets are meant to complement for that and provide practice needed to develop computational skills.