Classical & quantum integrable systems

Physics	Classical mechanics	Quantum mechanics
Maths	Poisson geometry	Representation theory

$$\text{Poisson-Lie groups} \underset{\text{classical limit}}{\overset{\text{quantisation}}{\rightleftarrows}} \text{Hopf algebras}$$

classical / quantum integrable systems

<u>Example</u>: Harmonic oscillator

$$\ddot{q} + q = 0 \Longleftarrow \begin{cases} \dot{q} = p \\ \dot{p} = -q \end{cases}$$

Let $\quad H(p, q, t) := \frac{1}{2}(p^2 + q^2)$

<u>Note</u>: 1) $\dfrac{dH}{dt} = p\dot{p} + q\dot{q} = -pq + pq = 0$

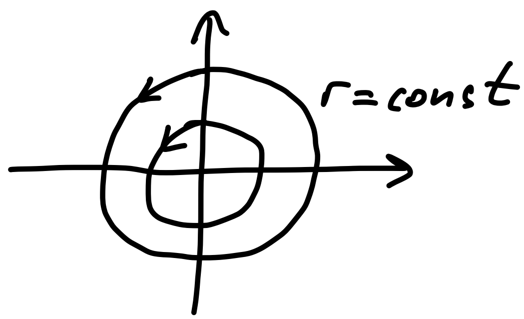
$$\Longrightarrow H = \text{const}$$

2) $\dot{q} = \dfrac{\partial H}{\partial p}, \quad \dot{p} = -\dfrac{\partial H}{\partial q}$

Parameterise: $\quad q = r\sin\theta$
$\qquad\qquad\qquad p = r\cos\theta$

$$\begin{cases}\dot q = p \\ \dot p = -q\end{cases} \iff \begin{cases}\dot r = 0 \\ \dot\theta = 1\end{cases} \implies \begin{array}{l} r(t) = r(0) \\ \theta(t) = \theta(0) + t\end{array}$$

exercise



Note: 1) $H = \frac{1}{2} r^2$

2) $dq \wedge dp = dH \wedge d\theta$

exercise

Poisson formalism:

$\mathcal{M} = \mathbb{R}^2$

$\{\cdot,\cdot\}$ — Lie Bracket on $Fun(\mathcal{M})$, satisfying $\{fg,h\} = f\{g,h\} + g\{f,h\}$

Poisson Bracket

Let $\{p,q\} = 1$

$$\{H,q\} = \frac{\partial H}{\partial p}\{p,q\} + \frac{\partial H}{\partial q}\{q,q\} = \frac{\partial H}{\partial p}$$

$$\{H,p\} = -\frac{\partial H}{\partial q}$$ ← exercise

Eq-n of motion: $\dot{f} = \{H,f\}$

Quantisation:

Heisenberg principle: $\hat{p}$ & $\hat{q}$ do <u>not</u> commute

$$[\hat{p}, \hat{q}] = \hbar$$

$\{\cdot, \cdot\} \longrightarrow [\cdot, \cdot]$ ← Lie Bracket

Eq-n of motion: $\dot{f} = \frac{i}{\hbar}[H, f]$ ← $\sqrt{-1}$

H, f – depend on $\hat{p}, \hat{q}$

Rep-n: $\mathbb{C}\langle \hat{p}, \hat{q} \rangle \circlearrowright L^2(x, dx)$

$\hat{q} \longmapsto x$ – multiplication

$\hat{p} \longmapsto \hbar \frac{\partial}{\partial x}$ – differentiation

Topics: Classical:

- Poisson-Lie groups
- Hamiltonian reduction
- classical r-matrices

Quantum:

- quantum R-matrices
- quantum groups
- Yangians

Etingof-Schiffman

Examples:

- Toda
- XXZ
- Gaudin

TBD

Prerequisites: Lie groups

Lie algebras

differential geometry