Analysis Theme: Pure and Applied

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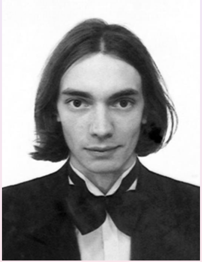
Outline

What is Analysis?

Mathematical Analysis is a far-reaching generalisation and abstraction of concepts you are familiar with from calculus and linear algebra. For example:

- **Measure theory** generalises intuitive notions of length, area, and volume;
- **Integration theory** uses measure theory to develop "better" integrals than the "usual" Riemann integral;
- **Metric spaces** and **normed spaces** develop the ideas of distance between vectors in $\mathbb{R}^n$ and of a magnitude of a vector, respectively, while the dot product in $\mathbb{R}^n$ gives rise to the idea of an **inner product** and **Hilbert spaces**;
- **Operator theory** and **Banach algebras** arise from an abstraction of properties of matrices.

Analysis has occupied the minds of some of the best mathematicians ever, such as Banach and Hilbert, and more recently (to mention just Fields medal winners, among many others) Schwartz, Grothendieck, J.-L. and P.-L. Lions and very recently, Villani and Figalli.



(a) Cèdric Villani

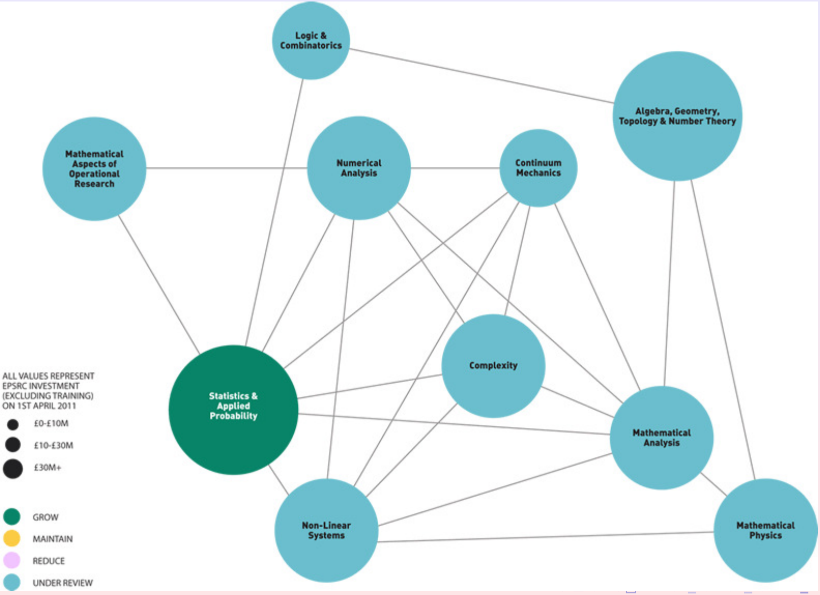
(b) Alessio Figalli

These generalisations provide a rigorous basis for most of mathematics applied to natural sciences: control, differential equations, calculus of variations (optimisation), economics, numerical analysis, mathematical physics, and probability theory.

It is an exciting time to be in Analysis! Major developments in recent years include **Optimal Transportation** and (the very mysterious) applications of **Convex Integration** techniques.

But what if I am not an Analyst? It is difficult to imagine an area of mathematics proper where the tools of Analysis are not used.

Interconnections: the EPSRC view



SMSTC Modules

2 pure modules:

- Measure and Integration (Semester I)
- Functional Analysis (Semester II)

2 applied modules:

- Dynamical Systems and Conservation Laws (Semester I)
- Elliptic and Parabolic PDEs (Semester II)

In pure mathematics the objects of interest are mathematical concepts and structures and relations among them, while in applied mathematics one uses tools developed in pure mathematics to illuminate problems coming from the natural sciences (physics, chemistry, geosciences, data science, biology, economics). Traffic here is not one way only!

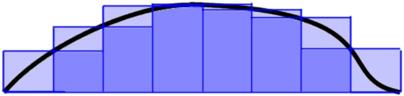
Measure and Integration

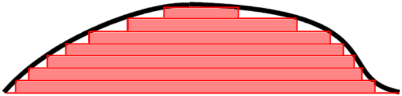
This course creates a rigorous setting for Probability Theory and introduces a very useful notion of integral, the Lebesgue integral. [Consider for $x \in [0,1]$,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q}. \end{cases}$$

Convince yourself that the Riemann integral of f is not defined!]

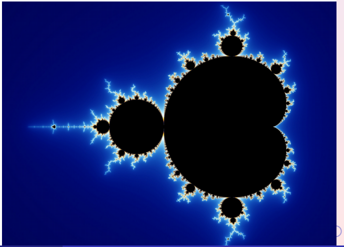
Riemann integral:
need to know measure of intervalls / cubes $[a,b]^n$





Lebesgue integral:
need to know measure of sublevel sets $\{x : f(x) < a\}$

sublevel sets can be wild:



Structure of Course

- ★ Riemann and Lebesgue integrals on $\mathbb{R}$;
- ★ $C(X)^*$ (the dual space of $C(X)$):

$$\Lambda \in C(X)^* \implies \Lambda(f) = \int_X f(x)d\mu(x)$$

- ★ Construction of Lebesgue measure on the real line;
- ★ Outer measures; Carathéodory construction of measure Hausdorff dimension.
- ★ Product measures: $X = Y \times Z$ (e.g. on $\mathbb{R}^2$); Fubini's theorem: when is it legitimate to change the order in a double integral.

The course will end with a discussion of Fourier series and overview of where research goes from here.

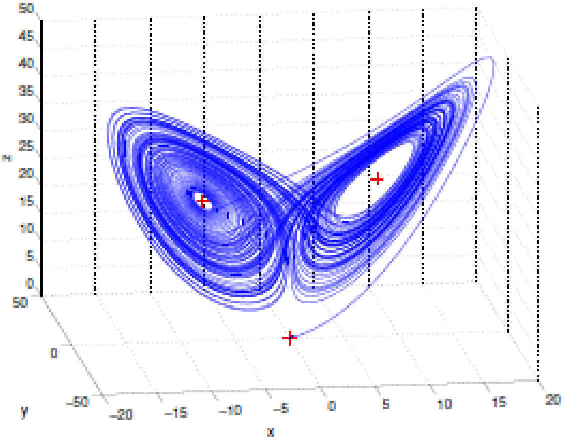
DSCL: Summary of the Course

- Dynamical systems and bifurcation theory;
- Scalar conservation laws: $u_t(x,t) + f(u(x,t))_x = 0$ and shock waves;
- Systems of hyperbolic PDEs.

Plus a guest lecture by MG, to introduce PDEs in general.

Note that concepts introduced in the discussion of ODEs are also applicable to evolutionary PDEs, integro-differential equations and delay-differential equations (“infinite-dimensional dynamical systems”); see also the supplementary course given by Prof. J. M. Ball.

Dynamical systems and conservation laws
Lorentz system of ODEs: weather forecasts



ODE Example

- Lorenz Equations

$$\left.\begin{array}{rcl}\frac{dx}{dt} & = & \sigma(y-x) \\ \frac{dy}{dt} & = & rx - y - xz \\ \frac{dz}{dt} & = & xy - bz\end{array}\right\} \qquad (1)$$

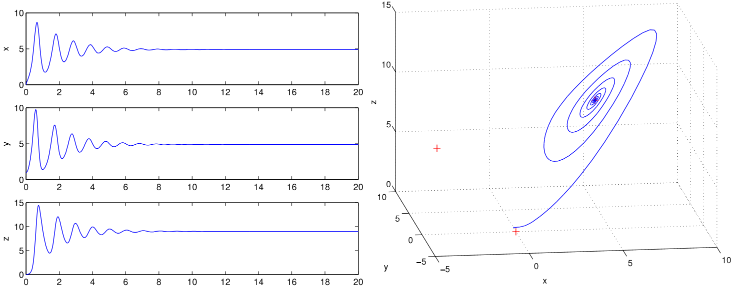
where σ, r, b are positive parameters.

- Simple looking system of ODEs derived by Lorenz to help analyse theoretical problems in meteorology and weather prediction.

Based on a simplified model of convection: when a fluid is heated from below.

Let's look at what happens to **same** initial data as r is increased.

Lorenz Equs: $r = 10$, $\sigma = 10, b = 8/3$

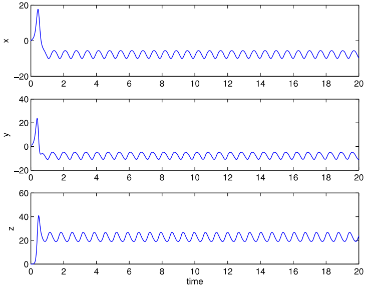


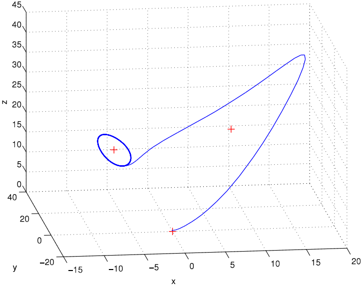
$x(t), y(t), z(t)$

Phase space $= \mathbf{R}^3$

Solution converges to a 'fixed point'.

Lorenz Equs: $r = 24.05$, $\sigma = 10, b = 8/3$



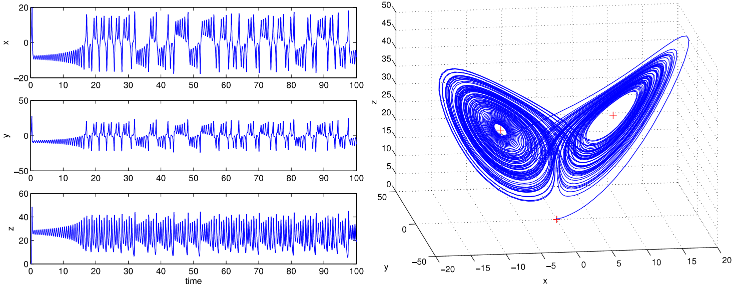


$x(t), y(t), z(t)$

Phase space $= \mathbf{R}^3$

Solution converges to a 'periodic orbit'.

Lorenz Equs: $r = 28$, $\sigma = 10, b = 8/3$



$x(t), y(t), z(t)$ Phase space $= \mathbf{R}^3$

Solution converges to classic 'chaotic attractor'.

FA: Basics (3 Lectures)

- Banach and Hilbert spaces; $L^p(X)$ spaces, $C(X)$, $C(X)^*$, etc ..., setting for Fourier series;
- Linear operators and linear functionals;
- Fundamental theorems: as in the 'Scottish book' from the famous café in Lwów: Baire Category, Open Mapping, Uniform Boundedness Principle, etc....

FA: Dual Spaces and Weak Topologies (2 Lectures)

- Weak and weak* topologies, Banach-Alaoglu Theorem, (weak* compactness of unit ball);
- Convexity in infinite dimensional setting: Krein-Milman Theorem (very useful convexity result); Haar measure on compact groups.

FA: Spectral Theory and Banach Algebras (2 Lectures)

- Compact operators and their spectra, spectral theorem for self-adjoint operators, general spectral theory;
- Banach algebras; commutative C^*-algebras – Gelfand's theory.

Again, the aim is to end with a survey of research topics.

Partial Differential Equations (PDEs): Basic Questions:

A PDE is an equation of the form $\mathcal{F}(x, y, \ldots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^2 u}{\partial x \partial y}, \ldots) = 0$, where $\mathcal{F}$ is known and $u(x, y, \ldots)$ is not. In this course we are interested in:

- Existence of solutions; uniqueness; dependence on data *(existence of classical solutions for Laplace and heat equations, Sobolev spaces and weak solutions)*;
- Qualitative properties of solutions *(self-similar solutions and travelling waves, maximum principles; Gidas-Ni-Nirenberg theorem on symmetry of solutions)*;
- Numerical approximation *(mathematical background behind finite elements [for implementation of numerical methods please choose other modules!])*.

PDEs are ubiquitous in the classical descriptions of natural phenomena:

- d'Alembert 1752, 1–dim. vibrating string equation:

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = f(t,x)$$

- Euler 1759, D. Bernoulli 1762, wave equation for acoustics:

$$\partial_t^2 u(t,\vec{x}) - \Delta u(t,\vec{x}) = f(t,\vec{x}) \ , \qquad \Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$$

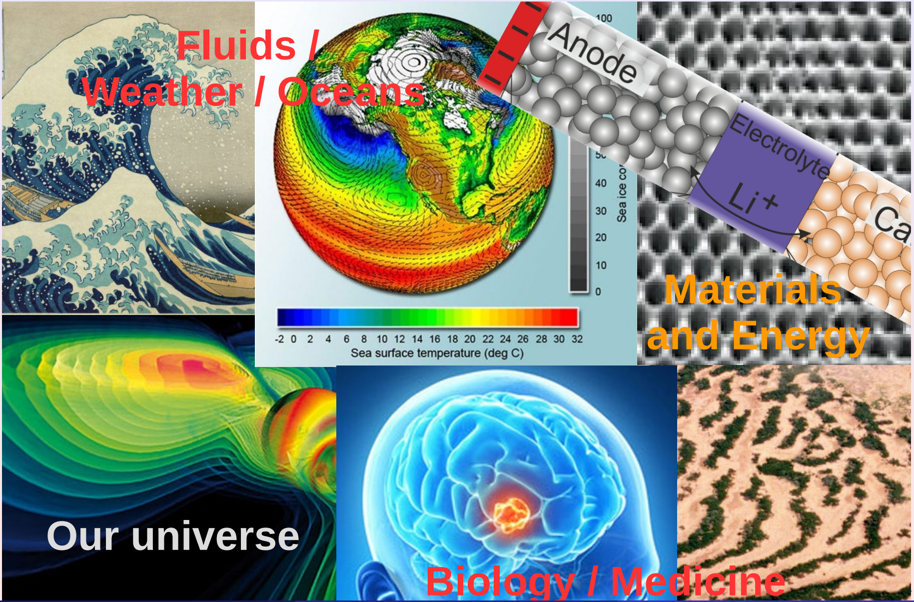
- Laplace 1780, Laplace equation for gravitational potential field:

$$-\Delta u(\vec{x}) = f(\vec{x})$$

- Fourier 1822, the heat equation:

$$\partial_t u(t,\vec{x}) - \Delta u(t,\vec{x}) = f(t,\vec{x})$$

Some Modern Applications of PDEs



Prerequisites for the pure courses

Undergraduate real analysis: Sequences, series, pointwise and uniform convergence, continuous and differentiable functions; basic properties; open, closed and compact sets, countabililty;

For Functional Analysis: If you have seen measure theory and some Fourier analysis before, great! Without knowledge of measure theory, should be able to get a lot out of the course, but need to ignore some of the examples.

Assessment for both courses

Ongoing feedback

- Exercises associated to each lecture.
- Not part of formal assessment; for discussion in tutorials.
- Some lecturers will ask for groups to present some of these in lectures.

Formal assessment for each course

- Two assignments for each course; normally consisting of three questions per assessment;
- In particular, there will be no measure theory knowledge assumed in the Functional Analysis assessment.

Prerequisites for the Applied Courses

- undergraduate ordinary differential equations;
- single- and multi-variable calculus, linear algebra;

Note: Semester II course is more or less self-contained, but Semester I course is useful for motivation and background.

Assessment for the Applied Courses

Ongoing feedback

- ✦ There are exercises associated to each lecture: not part of formal assessment but for discussion in tutorials.
- ✦ Some lecturers might ask for groups to present some of these in lectures.

Formal assessment for each course: Around 8 questions each semester, divided over 1 or 2 assignment sheets.