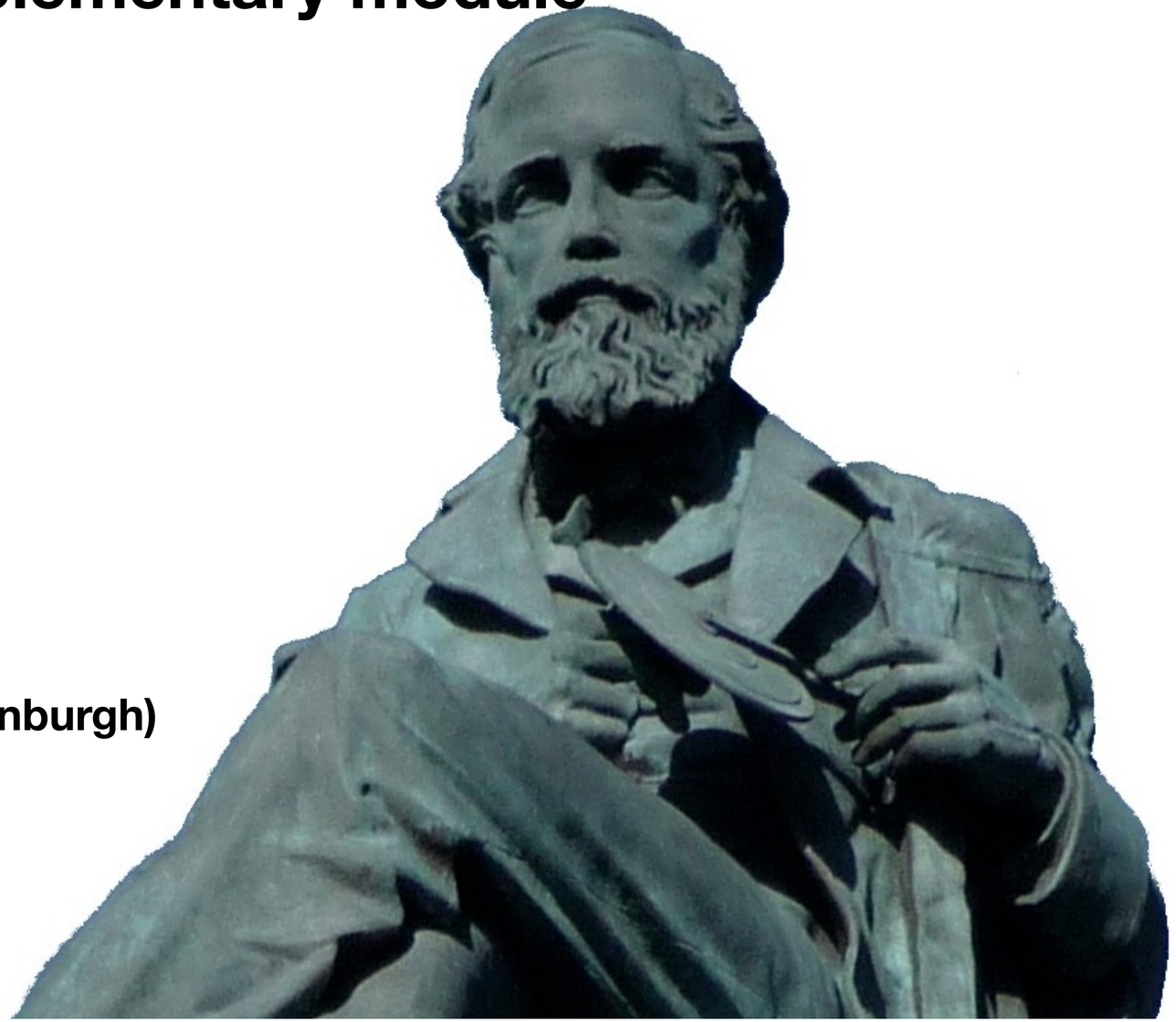
Geometry of Gauge Fields

A first-semester SMSTC supplementary module



José Figueroa-O'Farrill (The University of Edinburgh)
28 September 2022

Some history

James Clerk Maxwell (1831-1879)

Hermann Weyl (1885-1955)

1919. № 10.

ANNALEN DER PHYSIK.

VIERTE FOLGE. BAND 59.

1. *Eine neue Erweiterung der Relativitätstheorie;* *von H. Weyl.*

Kap. I. Geometrische Grundlage.

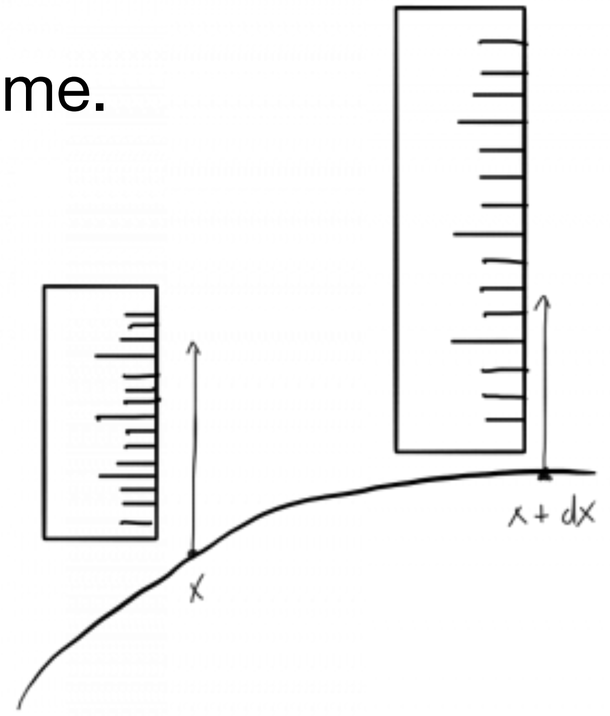
Einleitung. Um den physikalischen Zustand der Welt an einer Weltstelle durch Zahlen charakterisieren zu können, muß 1. die Umgebung dieser Stelle auf *Koordinaten* bezogen sein und müssen 2. gewisse *Maßeinheiten* festgelegt werden. Die bisherige Einsteinsche Relativitätstheorie bezieht sich nur auf den ersten Punkt, die Willkürlichkeit des Koordinatensystems; doch gilt es, eine ebenso prinzipielle Stellungnahme zu dem zweiten Punkt, der Willkürlichkeit der Maßeinheiten, zu gewinnen. Davon soll im folgenden die Rede sein.

Weyl's original gauge theory

Assume that length scale depends on position and time.

Let the length scale be given by a positive,

real-valued function $\ell : \mathbb{R}^4 \to \mathbb{R}^+$.



We may parallel transport a length scale ℓ in terms of a 1-form

$A = A_t dt + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$:

$$d\ell = -A\ell$$

Change the gauge $\ell' = \lambda\ell$ with a rescaling function $\lambda : \mathbb{R}^4 \to \mathbb{R}^+$. Then

$d\ell' = A'\ell'$ provided that A and A' are related by a **gauge transformation**:

$$A' = A - d\log\lambda$$

It follows that $F = dA$ is unchanged!

Weyl: electromagnetism?

Einstein: contradicts experiment!

Gauging the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\triangle\psi$$

Let $\psi' = e^{i\chi}\psi$. If χ is **constant**, ψ' also satisfies the Schrödinger equation.

Moreover, $|\psi'|^2 = |\psi|^2$. *What about if χ depends on position and time?*

In that case we would need to modify the equation.

Gauge invariance of Schödinger equation

(coupled to electromagnetism)

Let $D = d + iA$ and

$$i\hbar D_t \psi = -\frac{\hbar^2}{2m} \sum_{k=1}^{3} D_i^2 \psi$$

Now $\psi' = e^{i\chi}\psi$ solves the gauged Schrödinger equation provided that we also transform the electromagnetic potential:

$$A \mapsto A' = A - d\chi$$

What is gauge theory?

Gauge theory

- Redundancy in the formalism: the formalism depends on ‘gauges' but reality does not.
- Which mathematical quantities are gauge invariant?
- Much of fundamental physics is predicated on gauge theories: particle physics, general relativity,…
- Gauge theories used also in economics, finance, and of course mathematics, particularly topology (Donaldson, Chern-Simons,…)
- Gauge freedom captured by compact Lie groups such as $U(1)$, $SU(2)$,…

About this course

Geometry of Gauge Fields

Course content

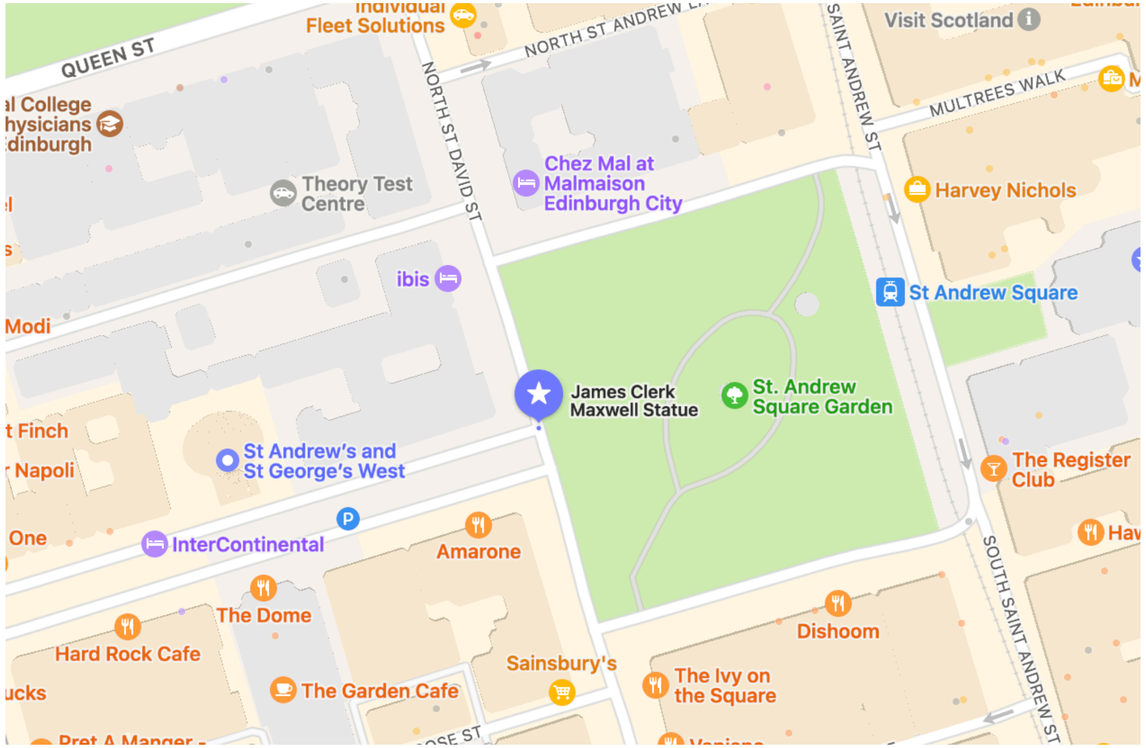
- Brief introduction to Lie groups and Lie algebras
- Principal bundles, associated bundles, connections, curvature, characteristic classes (Chern—Weil theory)
- Maxwell theory as a U(1) gauge theory and introduction to Yang—Mills theories.
- Gauge theories in 2 and 3 dimensions: Chern—Simons theory and the moduli space of flat connections on a Riemann surface.
- The standard model of particle physics, Seiberg—Witten theory.

Geometry of Gauge Fields

Administrative information

- Weekly lectures **Tuesdays 9:30-11** starting on **4 October 2022**
- Tutorials to be arranged locally (or online)
- Two assessments
- Team:
  - **Lotte Hollands** (Heriot—Watt)
  - **Brendan Owens** (Glasgow)
  - **Johan Martens** (Edinburgh)
  - **José Figueroa-O'Farrill** (Edinburgh)

A gauge theory giant



Michael Atiyah (1929-2019)