SMSTC Opening Symposium
28-29 September 2022

Infinite-dimensional dynamical systems

MAC-MIGS and SMSTC Advanced Course
Thursdays 10-12, starting 6 October
Bayes 5.46 and online

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Topics

The course will discuss methods for describing the dynamics of infinite-dimensional dynamical systems, for example generated by linear and nonlinear partial differential equations, and in particular their asymptotic behaviour as time $t \to \infty$.

Semiflows and approach to equilibrium. Definition of semiflows. Continuity properties. ω-limit sets, Lyapunov functions, LaSalle invariance principle, attractors, stability.

Linear semigroups and semilinear problems. Infinitesimal generators, Hille-Yosida theorem, weak solutions and the variation of constants formula. Semilinear problems treated via the variation of constants formula.

Applications to specific systems. Semilinear parabolic and hyperbolic PDE, Coagulation-fragmentation equations, viscoelasticity.

Example. Let $\Omega \subset \mathbb{R}^3$ be bounded, open, and consider the damped wave equation

$$u_{tt} + u_t - \Delta u + u^3 - u = 0,$$

with boundary condition $u|_{\partial\Omega} = 0$ and initial conditions

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x).$$

Formally we have the *energy equation*

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}u_t^2 + \frac{1}{4}(u^2 - 1)^2\right) dx = -\int_\Omega u_t^2 \, dx$$

so that

$$V(u, u_t) := \int_\Omega \left(\frac{1}{2}u_t^2 + \frac{1}{4}(u^2 - 1)^2\right) dx$$

is a *Lyapunov function*, i.e. it is nonincreasing along solutions and constant only at equilibrium solutions.

Questions.

1. Is this problem well-posed (i.e. existence and uniqueness of solutions depending continuously on the initial data $\{u_0, u_1\}$), and if so in what function space?

2. Does the energy equation hold?

3. Do solutions converge to equilibria, i.e. to solutions of

$$-\Delta u + u^3 - u = 0, \quad u|_{\partial\Omega} = 0$$

as $t \to \infty$?

4. Which equilibria are dynamically stable/unstable?

5. Is there a global attractor, and if so what is it?

Other problems with a similar structure.

The *semilinear heat equation*

$$u_t = \Delta u + f(u).$$

1D viscoelasticity of rate type

$$u_{tt} = S(u_x, u_{xt})_x.$$

The *discrete coagulation-fragmentation equations*

$$\dot{c}_j = \frac{1}{2}\sum_{k=1}^{j-1} a_{j-k,k}c_{j-k}c_k - b_{j-k}c_j) - \sum_{k=1}^{\infty}(a_{j,k}c_jc_k - b_{j,k}c_{j+k}),$$

$j = 1, 2, \ldots.$

plus perhaps other problems you may want to suggest ...

Prerequisites: Some knowledge of Sobolev spaces and elementary functional analysis an advantage

Assessment. Probably by 30 minute presentations of complementary topics.