Analysis theme: pure and applied

Heiko Gimperlein
h.gimperlein@hw.ac.uk

Heriot-Watt University

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Outline

2 pure modules:

- Measure and Integration (Semester I)
- Functional Analysis (Semester II)

2 applied modules:

- Dynamical Systems and Conservation Laws (Semester I)
- Elliptic and Parabolic PDEs (Semester II)

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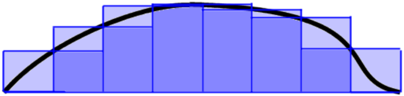
Integration as a tool

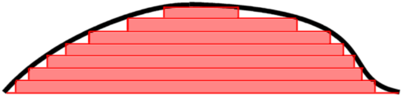
- Integration as a tool
  Geometry, representation theory, probability, applied analysis etc.
- Geometry – Gauss Bonnet Theorem

$$\int_M K \mathrm{d}A = 2\pi \, \chi(M)$$

- Applied analysis – solving differential equations
- Representation theory: average over compact group by integrating over the group.

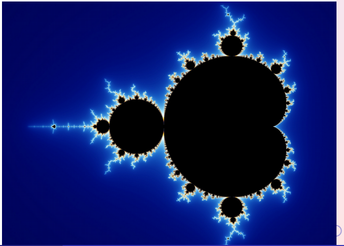
Riemann integral:
need to know measure of intervalls / cubes $[a,b]^n$



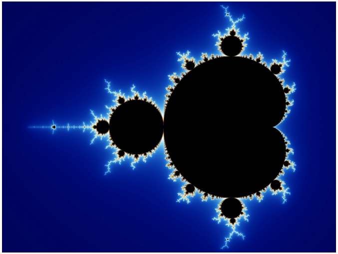


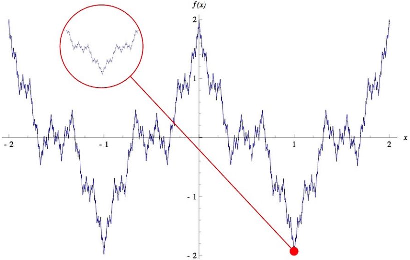
Lebesgue integral:
need to know measure of sublevel sets $\{x : f(x) < a\}$

sublevel sets can be wild:

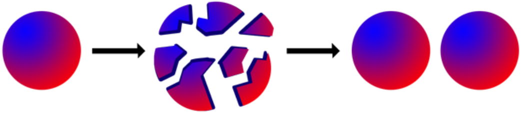


Sublevel sets can be wild
Can we define their volume?





There exist subsets of R^n for which volume can‘t be defined
Banach-Tarski paradox decomposes ball into two balls, using translations and rotations (using bizarre pieces)



→ Measure theory and integration

Abstract Measure and Integration

- Abstract measure spaces and Lebesgue integral
- Setting for Probability theory
  - A *measurable* set E corresponds to an **event**
  - A *measurable* function corresponds to a random variable
- Two lectures (foundations and complex measures)

This abstract framework will unify:

- Riemann and Lebesgue integral on $\mathbb{R}$: $\int_{\mathbb{R}} f(x)\,dx$
- Infinite series: $\sum_{n=1}^{\infty} a_n$
- Elements of $C(X)^*$ (the continuous dual space of $C(X)$):

$$\Lambda \in C(X)^* \quad \Longrightarrow \quad \Lambda(f) \; = \; \int_X f(x) d\mu(x)$$

Constructing measures

- Construction of Lebesgue measure on real line
- Carathéodory construction.
  - using outer measures
  - fractal sets and Hausdorff dimensions
- Product measures: $X = Y \times Z$ (e.g. $\mathbb{R}^2$
  - Fubini's theorem: when is it legitimate to change the order in a double integral.
- Radon measures – dual space of $C(X)$
- Total of 4 lectures (actually we'll construct Lebesgue measure on $\mathbb{R}$ before developing the abstract framework).
- End with discussion of Fourier series and overview of where research goes from here.

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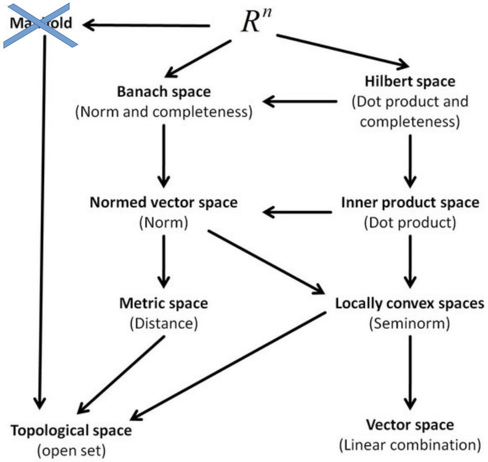
Functional analysis:
Vector spaces and operators in infinite dimensions



Functional Analysis

Provides tools used right across mathematics: Language of analysis

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I'm not an analyst, why should I care?

- Banach algebras in algebraic geometry (perfectoid spaces, Fields medal Peter Scholze 2018)
- Hilbert and Banach spaces in numerical finite element methods
- quantum mechanics = Hilbert spaces, spectral theory, C^* algebras
- ...

Basics

- Banach and Hilbert spaces

  $L^p(X)$ spaces, $C(X)$, $C(X)^*$, etc ...
  Setting for Fourier series
- Linear operators and linear functionals
- Fundamental theorems: in the 'Scottish book' from cafe in Lwów.

  Baire Category, Open Mapping, Uniform Boundedness Principle, etc...
- Three lectures

Dual spaces and weak topologies

- Weak and weak* topologies
  - Banach-Alaoglu (weak* compactness of unit ball)
- Convexity in infinite dimensional setting.
  - Krein-Milman Theorem (very useful convexity result)
  - Use to produce Haar measure on compact groups.
- Two lectures

Spectral Theory

- Compact operators and their spectra
- General spectral theory

  Banach algebras

  Spectral theorem for self-adjoint operators
- Commutative C^*-algebras – Gelfand's theory
- Four lectures; again aiming to finish with overview lecture pointing towards current research in this direction.

Prerequisites for both pure courses

Formal prerequisites for both pure courses

- Undergraduate real analysis
  - Sequences, series, pointwise and uniform convergence.
  - Continuous and differentiable functions, basic properties.
- Metric space topology (at least in $\mathbb{R}^d$)
  - continuity of functions, open, closed and compact sets
- Countablilty
  - A set S is **countable** if $S = \{s_1, s_2, s_3, \ldots\}$.
  - The set of reals $\mathbb{R}$ is **uncountable**!

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For: Measure and Integration

- Will start with full account of Riemann integral, so self contained.
- If you've not seen the Riemann integral (or a version of the Lebesgue integral on $\mathbb{R}$) before, you'll need to be motivated.

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For: Functional Analysis

- If seen measure theory and some Fourier analysis before ideal!
- Without measure theory, should be able to get a lot out of the course, but need to ignore some of the examples.

Assessment for both courses

Ongoing feedback

- Exercises associated to each lecture.
- Not part of formal assessment; for discussion in tutorials.
- Some lectures (certainly including me) will ask for groups to present some of these in lectures.

Formal assessment for each course

- Two assignments for each course; normally consisting of three questions per assessment.
- In particular, there'll be no measure theory assumed in the functional analysis assessment

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Partial differential equations (PDE)

Study equations $\mathcal{F}(x, y, \ldots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^2 u}{\partial x \partial y}, \ldots) = 0$

for a function $u = u(x, y, \ldots)$, where $\mathcal{F}$ is known and u is not.

Classical descriptions of natural phenomena:

- d'Alembert 1752, 1–dim. wave equation for vibrating string:

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = f(t,x)$$

- Euler 1759, D. Bernoulli 1762, wave equation for acoustics:

$$\partial_t^2 u(t,\vec{x}) - \Delta u(t,\vec{x}) = f(t,\vec{x}) \ , \qquad \Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$$

- Laplace 1780, Laplace equation for gravitational potential field

$$-\Delta u(\vec{x}) = f(\vec{x})$$

- Fourier 1822, heat equation

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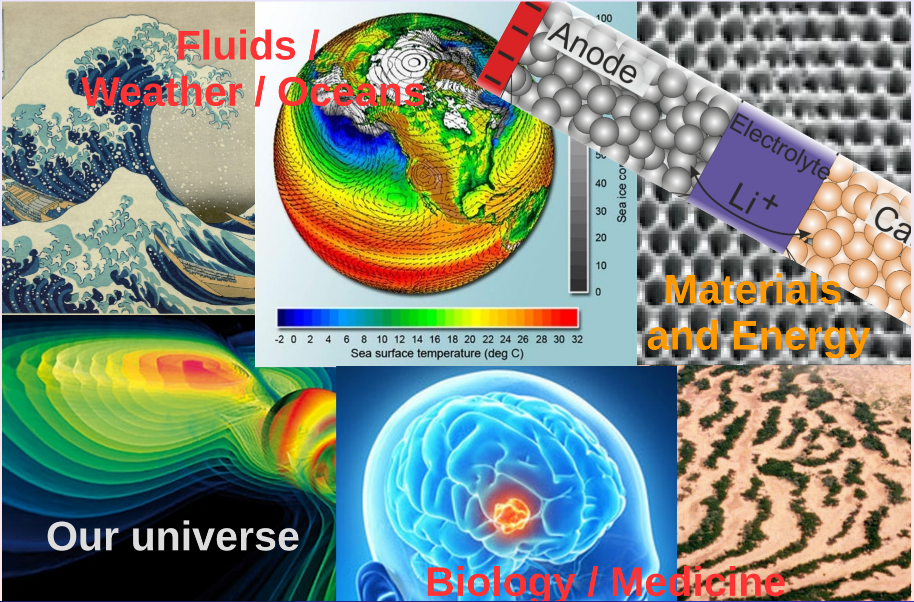
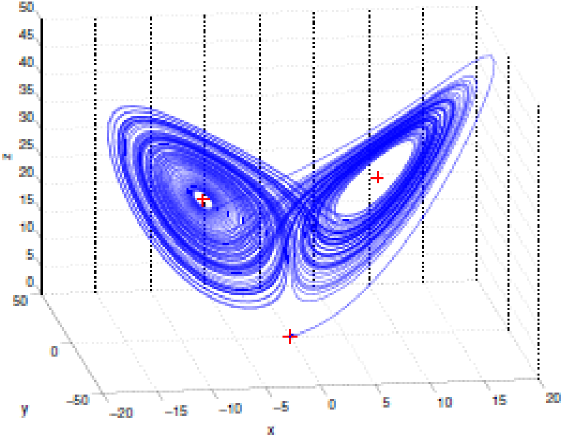
Some modern applications



Dynamical systems and conservation laws

Lorentz system of ODEs: weather forecasts



ODE Example

- Lorenz Equations

$$\left.\begin{array}{rcl} \frac{dx}{dt} & = & \sigma(y-x) \\ \frac{dy}{dt} & = & rx - y - xz \\ \frac{dz}{dt} & = & xy - bz \end{array}\right\} \qquad (1)$$

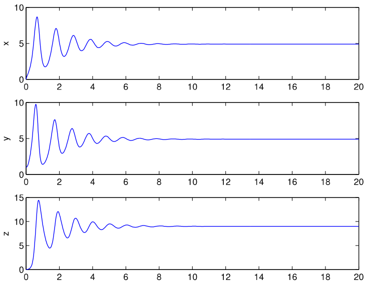
where σ, r, b are positive parameters.

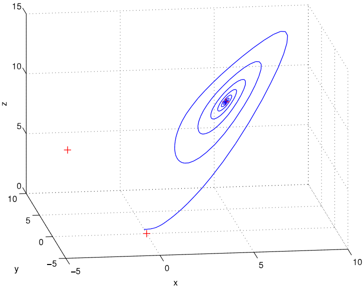
- Simple looking system of ODEs derived by Lorenz to help analyse theoretical problems in meteorology and weather prediction.

Based on a simplified model of convection: when a fluid is heated from below.

Let's look at what happens to **same** initial data as r is increased.

Lorenz Equs: $r = 10$, $\sigma = 10, b = 8/3$



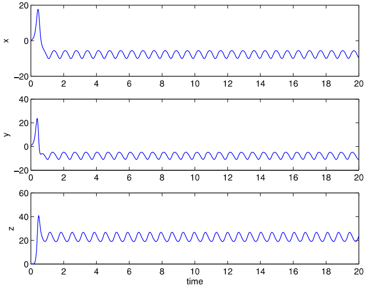


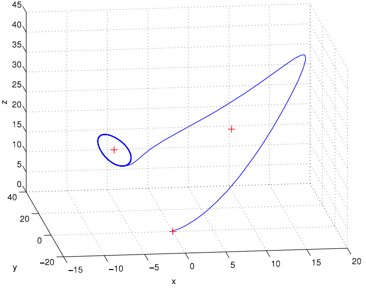
$x(t), y(t), z(t)$

Phase space $= \mathbf{R}^3$

Solution converges to a 'fixed point'.

Lorenz Equs: $r = 24.05$, $\sigma = 10, b = 8/3$



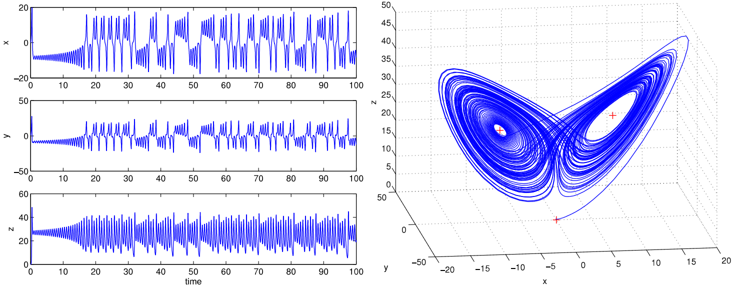


$x(t), y(t), z(t)$

Phase space $= \mathbf{R}^3$

Solution converges to a 'periodic orbit'.

Lorenz Equs: $r = 28$, $\sigma = 10, b = 8/3$



$x(t), y(t), z(t)$ Phase space $= \mathbf{R}^3$

Solution converges to classic 'chaotic attractor'.

Summary: Dynamical systems and conservation laws

- Dynamical systems and bifurcation
- Scalar conservation laws
- Systems of hyperbolic PDEs and shock waves

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Some basic questions

- Modelling:
  What are the equations and where do they come from?
  choose other modules
- $\exists$ solution? Uniqueness? Dependence on data? *existence of classical solutions for Laplace and heat equations (applied treatment)*
  Sobolev spaces and generalized weak solutions
- Qualitative properties?
  self-similar solutions and travelling waves, maximum principles, Gidas-Ni-Nirenberg theorem on symmetry of solutions
- numerical approximation *mathematical background behind finite elements, for numerical methods choose other modules*

Formal prerequisites for both applied courses

- undergraduate ordinary differential equations
- single- and multivariable real analysis, linear algebra
- Semester II more or less self-contained, but Semester I useful for motivation and background.

Assessment for both applied courses

Ongoing feedback

- Exercises associated to each lecture.
- Not part of formal assessment; for discussion in tutorials.
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Formal assessment for each course

- around 8 questions each semester, divided over 1 or 2 assignment sheets.

Conclusion

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Interactions: Pure maths is being applied, applied math solves pure problems.