SMSTC Supplementary Module: Geometry of Gauge Fields

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What is this module about?

The inventors of gauge theory

Figure: James Clerk Maxwell and Hermann Weyl

A fruitful failure

1919. № 10.

ANNALEN DER PHYSIK.
VIERTE FOLGE. BAND 59.

1. *Eine neue Erweiterung der Relativitätstheorie;*
von H. Weyl.

Kap. I. Geometrische Grundlage.

Einleitung. Um den physikalischen Zustand der Welt an einer Weltstelle durch Zahlen charakterisieren zu können, muß 1. die Umgebung dieser Stelle auf *Koordinaten* bezogen sein und müssen 2. gewisse *Maßeinheiten* festgelegt werden. Die bisherige Einsteinsche Relativitätstheorie bezieht sich nur auf den ersten Punkt, die Willkürlichkeit des Koordinatensystems; doch gilt es, eine ebenso prinzipielle Stellungnahme zu dem zweiten Punkt, der Willkürlichkeit der Maßeinheiten, zu gewinnen. Davon soll im folgenden die Rede sein.

Die Welt ist ein vierdimensionales Kontinuum und läßt sich deshalb auf vier Koordinaten $x_0\, x_1\, x_2\, x_3$ beziehen. Der Übergang zu einem anderen Koordinatensystem $\bar{x}_i$ wird durch stetige Transformationsformeln

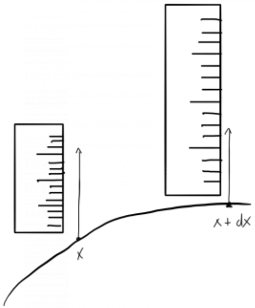
$$x_i = f_i(\bar{x}_0\, \bar{x}_1\, \bar{x}_2\, \bar{x}_3) \qquad (i = 0, 1, 2, 3) \tag{1}$$

vermittelt. An sich ist unter den verschiedenen möglichen Koordinatensystemen keines ausgezeichnet. Die Relativkoordinaten dx_i eines zu dem Punkte $P = (x_i)$ unendlich benachbarten $P' = (x_i + dx_i)$ sind die Komponenten der infinitesimalen Verschiebung $\overrightarrow{PP'}$ (eines „Linienelementes" in P). Sie transformieren sich beim Übergang (1) zu einem anderen Koordinatensystem $\bar{x}_i$ linear:

$$d x_i = \sum_k \alpha_k^i \, d \bar{x}_k ; \tag{2}$$

α_k^i sind die Werte der Ableitungen $\partial f_i / \partial \bar{x}_k$ im Punkte P. In der gleichen Weise transformieren sich die Komponenten ξ^i irgendeines *Vektors* in P. Mit einem die Umgebung von P bedeckenden Koordinatensystem ist ein „Achsenkreuz" in P

Length scale (‘gauge’) depends on position and time?



Assume the length scale is given by a positive, real-valued function $\ell : \mathbb{R}^4 \to \mathbb{R}^+$.

Length scale ('gauge') depends on position and time?

Parallel transport of a length scale in terms of 1-form $A = A_t dt + A_1 dx_1 + A_2 dx_2 + A_3 dx_3$:

$$d\ell = -A\ell, \qquad (1)$$

Change the gauge $\ell' = \lambda\ell$ with re-scaling function $\lambda : \mathbb{R}^4 \to \mathbb{R}^+$. In order to maintain the condition (1) in the new gauge we require

$$A' = A - d\ln\lambda.$$

$F = dA$ is unchanged!
Electromagnetic field? Einstein: ruled out by experiment

Wavefunction of free particle non-relativistic quantum mechanics is a map $\psi : \mathbb{R}^4 \to \mathbb{C}$ which obeys

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi. \tag{2}$$

Normalised $\int_{\mathbb{R}^3} |\psi(t, \boldsymbol{x})|^2 d^3\boldsymbol{x} = 1, \quad \forall t \in \mathbb{R}$, so that the probability of the particle being in a region $R \subset \mathbb{R}^3$ at time t is

$$p(t, R) = \int_R |\psi(t, \boldsymbol{x})|^2 d^3\boldsymbol{x}.$$

The probability is invariant under a 'phase change'

$$\psi \mapsto \psi' = e^{i\chi}\psi, \quad \chi : \mathbb{R}^4 \to [0, 2\pi).$$

Gauging the Schrödinger Equation

Introduce **gauge potential** a on $\mathbb{R}^4$ and **covariant derivatives**

$$D = d + A$$

Then the **gauged** Schrödinger equation

$$i\hbar D_t \psi = -\frac{\hbar^2}{2m} \sum_{j=1}^{3} D_j^2 \psi, \qquad (3)$$

is **covariant** under **gauge transformation**

$$\psi \mapsto \psi' = e^{i\chi}\psi, \quad a \mapsto a - d\chi.$$

This is the gauge potential of **Maxwell's electrodynamics**!

What is gauge theory?

- All measurements depend on conventions and ‘gauges’ - but reality does not. Which mathematical quantities are gauge invariant?
- Gauge theories now used in physics, mathematics, economics and finance.
- The unreasonable effectiveness of gauge theories in modern physics and mathematics. Why?
- Here: gauge freedom captured by compact Lie groups $U(1), SU(2), SU(n)$....

Contents

1. Review of vector fields and differential forms on manifolds, introduction to Lie groups and Lie algebras.
2. Fibre bundles and associated vector bundles, connections, curvature, characteristic classes;
3. Maxwell theory as U(1) gauge theory, Dirac monopole as curvature, wave function of charged particle as section of associated line bundle;
4. Chern-Simons theory and the moduli space of flat connections on a Riemann surface, Atiyah-Bott symplectic structure;
5. Classical Yang-Mills theory, monopoles and instantons, self-duality equations, ADHMN construction of instantons and monopoles;
6. Outlook on moduli spaces of instantons and monopoles, S-duality and L^2-cohomology.

A giant of modern gauge theory

Figure: Michael Atiyah with statue of James Clerk Maxwell in Edinburgh

Bibliography

1. G. L. Naber, *Topology, Geometry and Gauge Fields: Foundations*. Springer, New York, 2011.
2. G. L. Naber, *Topology, Geometry and Gauge Fields: Interactions*. Springer, New York, 2011.
3. M. F. Atiyah, *The Geometry of Yang-Mills Fields*, Publication of the Scuola Normale, Pisa, 1979.
4. M. F. Atiyah, *The Geometry and Physics of Knots*, Cambridge University Press, Cambridge 1990.

Who should take this module?

Prerequisites

- Some understanding of differentiable manifolds and differential forms, group theory
- Some familiarity with Lie algebras and Lie groups would be helpful but will **not** be assumed.

Take this course if you are interested in ...

- Connections between geometry, topology and physics,
- The mathematical theory of fibre bundles and connections,
- The language in which the Standard Model of Particle Physics is formulated,
- Beautiful applications of mathematics to physics: Yang-Mills theory, magnetic monopoles, instantons.
- Surprising applications of physics to mathematics: Donaldson theory, knot invariants from Chern-Simons theory, Seiberg-Witten theory (but we will not study them here in any detail!).

Who is teaching this module?

The team

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