SMSTC, Structure and Symmetry

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Structure & Symmetry

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Wikipedia tells us:

- **Algebra** (from Arabic "al-jabr", literally meaning "reunion of broken parts") is the study of mathematical symbols and the rules for manipulating these symbols
- **Geometry** (from the Ancient Greek: geo- "earth", -metron "measurement") is a branch of mathematics concerned with questions of shape, size, relative position of figures, and the properties of space
- **Topology** (from the Greek topos, place, and logos, study) is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing

Stream overview

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Semester 1

★ **Groups, Rings and Modules**

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★ **Algebraic Topology**

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Stream overview

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Semester 2

★ **Algebras and Representation Theory**

- Sira Gratz, Glasgow
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- Dougal Davis, Edinburgh
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★ **Manifolds**

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Prerequisites

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- **Groups, Rings and Modules**
  - Basic linear algebra and basic algebra concepts.
    - Definitions and examples of groups, rings and fields.
  - Basic notions of group theory.
    - Lagrange's theorem, normal subgroups and factor groups.

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  - Basic linear algebra and basic algebra concepts.
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- **Algebraic Topology**
  - A course in metric spaces or topological spaces.
  - A course in group theory.
    - Group actions.
    - Finitely generated abelian groups.

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★ **Algebras and Representation Theory**

- The notion of a module and related concepts.
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  - The notion of a module and related concepts.
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  - Some commutative algebra.
- **Manifolds**
  - Standard calculus courses.
    - Green's theorem.
  - Basic courses in linear algebra.
    - Abstract vector space.

Groups, Rings and Modules

Groups, Rings and Modules

- Groups.
  1. Simple groups, Jordan-Holder Theorem, (semi)direct products.
  2. Permutation representations and group actions.
  3. Sylow Thorems and applications.
  4. Abelian, soluble and nilpotent groups.
  5. Free groups and presentations.

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- Commutative rings.
  1. Modules: introduction.
  2. Chain conditions and Hilbert's basis theorem.
  3. Fields and numbers.
  4. Affine algebraic geometry.
  5. Hilbert's Nullstellensatz.

Representation Theory

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- Noncommutative rings.
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  - The Artin-Wedderburn Theorem.

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  - Finitely generated modules over principal ideal domains.
  - The Artin-Wedderburn Theorem.
- Representation Theory.
  - Representations and characters.
  - Orthogonality relations.
  - Induced representations.
  - Computing character tables.

Algebraic Topology

Algebraic Topology

(1) Basic examples and constructions of topological spaces.
(2) Manifolds, basic homotopy theory and homotopy groups.
(3) Cofibrations, cell attachments and CW-complexes.
(4) Cellular approximation and relative homotopy groups.
(5) Fibre bundles, fibrations and the Hopf map.
(6) An introduction to homology.
(7) Homotopy invariance, exactness and excision.
(8) Computations and applications of homology.
(9) An introduction to cohomology.

Manifolds

Manifolds

(**1**) Implicit Function and Sard's Theorems, abstract manifolds.
(**2**) Tangent vectors and the tangent bundle, vector bundless.
(**3**) Vector fields and flows, Lie derivative, the Frobenius Theorem.
(**4**) Differential forms, Stokes' Theorem and Poincare duality.
(**5**) Riemannian metrics, connections, the Levi-Civita connection.
(**6**) Geodesics, the exponential map.
(**7**) Curvature and integrability, Riemannian curvature.
(**8**) Gauss Formula and the Theorema Egregium.
(**9**) Euler characteristic, the Gauss-Bonnet Theorem for surfaces.

Lüroth Problem

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Are there any other options for the subfield $\mathbb{F}$?

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Theorem (Lüroth)

NO.

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- One can choose generators of $\mathbb{F}$ such that Σ is *very good*.

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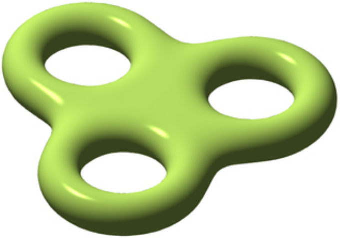
Example

If Σ is given by $x^3y + y^3 + x = 0$ in $\mathbb{C}^2$, then **S** looks like



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If it is one-to-one, then we are done.

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$$\begin{aligned} 2 = \chi\big(S^2\big) = \chi\big(S^2 \setminus \nabla\big) + \chi\big(\nabla\big) = \chi\big(S^2 \setminus \nabla\big) + \big|\nabla\big| = \\ = d\chi\big(\mathbf{S} \setminus \Delta\big) + \big|\nabla\big| \leqslant d\chi\big(\mathbf{S} \setminus \Delta\big) + d\big|\Delta\big| = d\chi\big(\mathbf{S}\big) = d\big(2 - 2\mathbf{g}\big), \end{aligned}$$

which implies that $\mathbf{g} = 0$.