SMSTC Supplementary Module: Classical and Quantum Integrable Systems

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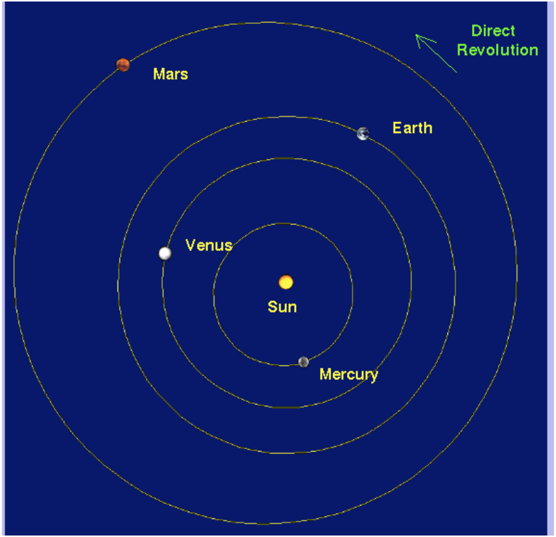
What is this module about?

The oldest classical integrable system: Kepler's problem



Classical integrability in a nutshell

We have

- A symplectic manifold M of dimension $2n$: physically the phase space or space of 'positions and momenta'
- Poisson brackets $\{f, g\}$ for $f, g \in C^\infty(M)$
- A Hamiltonian $H \in C^\infty(M)$ which generates time evolution according to
$$\dot{f} = \{H, f\}.$$

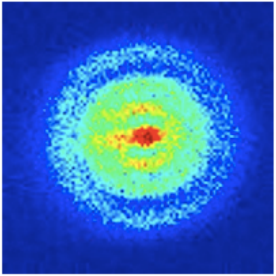
We want:

- $n - 1$ further functions $f_1, \ldots, f_{n-1}$ so that
$$\{H, f_i\} = 0, \quad \text{and} \quad \{f_i, f_j\} = 0, \quad i, j = 1, \ldots, (n-1).$$
- An understanding of the geometrical flow generated by H.

Contents

Quantum integrability: the hydrogen atom

Orbital structure of H excited state



Quantum integrability in a nutshell

We have

- A Hilbert space $\mathcal{H}$: the space of quantum states.
- Self-adjoint operators $A : \mathcal{H} \to \mathcal{H}$: the observables of the theory.
- A particular self-adjoint operator, called the Hamiltonian H, which generates time evolution according to

$$\dot{A} = i\hbar[H, A].$$

We want:

- A (possibly infinite) family operators A_i, $i = 1, 2 \ldots$ which satisfy

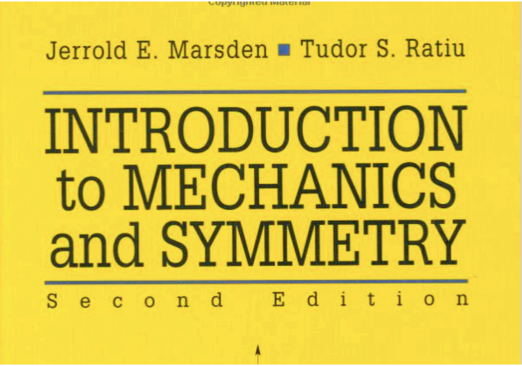
$$[A_i, A_j] = [A_i, H] = 0, \quad i, j = 1, \ldots$$

- The ground state of H
- The discrete spectrum of H and the A_i, $i = 1, 2 \ldots$
- Other quantum properties of the system such as correlation functions.

Contents

1. **Statistical lattice models & combinatorics:** Motivating examples from enumerative combinatorics; partition functions; transfer matrices; quantum R-matrices; the quantum Yang-Baxter equation;
2. **The Yang-Baxter Algebra & the Bethe Ansatz:** Monodromy matrices and the Yang-Baxter algebra; diagonalising the transfer matrix; Bethe equations; guiding example: the 6-vertex model on the cylinder and the torus
3. **Quantum Groups I:** Axioms and examples; the Yang-Baxter algebra as a quantum group; the 6-vertex model and XXZ spin-chain as motivating example
4. **Quantum Groups II:** Representation theory; $U_q(\widehat{\mathfrak{sl}}_2)$; the R-matrix; short-exact-sequences and the XXZ fusion hierarchy
5. **Quantum integrability at the boundary:** The reflection equation and K-matrices; examples of diagonal and non-diagonal boundary conditions; the XXZ Hamiltonian and Bethe equations in the presence of boundaries;

Bibliography



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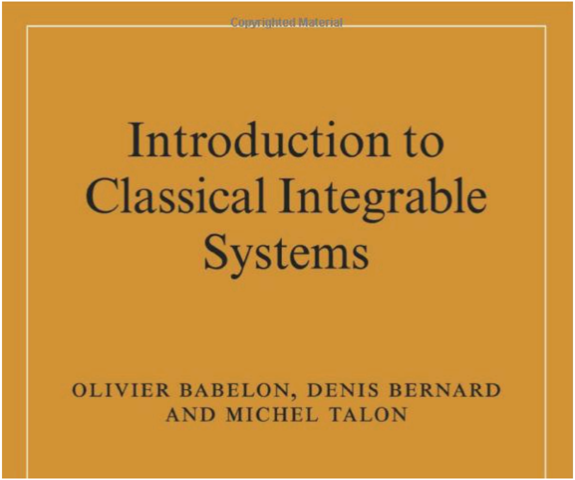
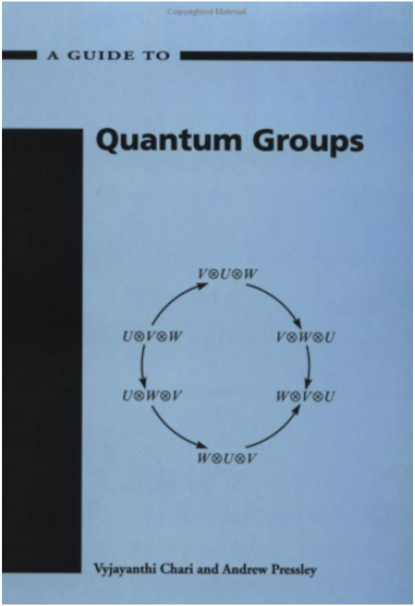


Bibliography



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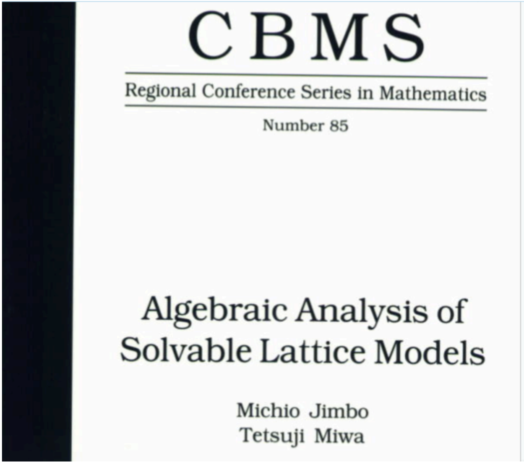

Bibliography

arXiv.org > math-ph > arXiv:1010.5031

Mathematical Physics

Lectures on the integrability of the 6-vertex model

N. Reshetikhin

(Submitted on 25 Oct 2010)

This is an overview of various aspects of the 6-vertex model in statistical mechanics and related models.

Comments:	81 pages. Lectures given at the summer school in theoretical physics, Les Houches, July 2008. Published in Les Houches Proceedings
Subjects:	**Mathematical Physics (math-ph)**; Statistical Mechanics (cond-mat.stat-mech)
MSC classes:	82B20, 82B23
Cite as:	arXiv:1010.5031 **[math-ph]**
	(or arXiv:1010.5031v1 **[math-ph]** for this version)

Who should take this module?

Prerequisites

- Some understanding of differentiable manifolds and differential forms, group theory
- Some familiarity with Lie algebras and Lie groups would be helpful but will **not** be assumed.

Take this course if you are interested in ...

- A rapidly evolving field which connects algebra, geometry, topology and physics
- A geometrical formulation of classical mechanics in the language of symplectic and Poisson structures
- The classical Yang-Baxter equation and how it gives rise to Poisson-Lie structures and integrable systems
- The role of quantum groups in quantum integrable systems
- A toolkit for solving problems in quantum field theory, string theory and condensed matter physics which are not (easily) amenable to other techniques

Who is teaching this module?

The team

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