Supplementary modules

What's on offer

Almost all of this year's supplementary modules are provided by MIGSAA...

...all except one!

That's

The mod 2 Steenrod algebra in theory and in practice

taught by Andy Baker (Glasgow), starting in week 3 of this semester.

The mod 2 Steenrod Algebra in Theory and in Practice

The mod 2 Steenrod Algebra $\mathcal{A}$ is the ring of operations (natural endomorphisms) of mod 2 cohomology $H^*(-;\mathbb{F}_2)$.

Topics:

1) The Steenrod operations, the Steenrod algebra and its dual. Algebraic structure, finite dimensional sub Hopf algebras including the $\mathcal{A}(n)$.

2) Applications in stable and unstable homotopy theory.

3) The Steenrod algebra in the wider world: group cohomology and invariant theory,

4) Deeper structure of modules over $\mathcal{A}$ and $\mathcal{A}(n)$. Stable module category of $\mathcal{A}(1)$.

Prerequisites: Suitable for anyone with basic knowledge of cohomology with coefficients for spaces and who wishes to learn about one of the central tools of modern homotopy theory. Algebraic aspects might interest mathematicians working on representation theory of finite dimensional algebras, cohomology of finite groups and Hopf algebras.

Practicalities: May be an opportunity for PhD students and others to give short talks on topics that particularly interest them – this can be arranged during the course.

Maxwell Institute Graduate School in Analysis and its Applications – Advanced Courses 2017-18 available through SMSTC

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Outline

1. Semester 1 courses
   - Advanced PDE I: Elliptic and parabolic PDE
   - Analysis and Numerics of Stochastic PDEs
   - A Fourier Based Approach of (stochastic) Integration
2. Semester 2 courses
   - Analysis of Diffusion Processes
   - What is ... Numerical Analysis?
   - Advanced PDE II: Hyperbolic PDEs
   - Homogenisation I
   - Homogenisation II

Advanced PDE I: Elliptic and parabolic PDE

1) Examples of elliptic equations, maximum principles (strong, weak), Hopf's Lemma, comparison principle.

2) Classical solutions, Bernstein estimate, applications.

3) Schauder estimates

4) Approximation by smooth functions, Sobolev spaces, embeddings, traces

5) Weak solutions, Lax-Milgram

6) Interior regularity, Boundary regularity

7) Parabolic equations, main examples, maximum principle

8) Parabolic setting and Sobolev spaces

9) Global in time solutions for nonlinear parabolic equations with small initial date

10) Energy estimates

Advanced PDE I: Elliptic and parabolic PDE, cont'd

Prerequisites:

1) rigorous multivariable calculus (continuity, differentiability, chain rule, integration)

2) Metric spaces, Banach spaces, Hilbert space, weak/strong convergence

3) vector calculus, Green's formula, (normal, tangent/vectors, parametrisation of surfaces and curves.)

Book: L.C. Evans, Partial Differential Equations, AMS Graduate Studies in Mathematics.

Analysis and Numerics of Stochastic PDEs

In the first part of the lectures basic methods of solving stochastic partial differential equations (SPDEs) of parabolic type will be presented. In particular, main results of the L^2 and L^p theories for SPDEs given in the whole Euclidean space will be summarised, and the theory of SPDEs given on domains will be presented in more details.

In the second part of the lectures methods for solving SPDEs numerically will be studied, theorems on accuracy of numerical schemes will be proved. Applications from population genetics and stochastic filtering will be discussed.

A Fourier Based Approach of (stochastic) Integration and Applications

In 1961, Ciesielski established a remarkable isomorphism of spaces of Hölder continuous functions and Banach spaces of real valued sequences. The isomorphism can be established along Fourier type expansions of Hölder continuous functions using the Haar-Schauder wavelet. We will start with Schauder representations for a pathwise approach of the integral of one function with respect to another one, using Ciesielski's isomorphism. We cover the paradigm of Young integral and the rough paths integral of T. Lyons. Our approach allows understanding this more involved theory of integration, purely from an analytical perspective using Paley-Littlewood decompositions of distributions, and Bony paraproducts in Besov spaces.

A Fourier Based Approach of (stochastic) Integration and Applications, cont'd

We apply the theory within a probabilistic framework and express Brownian motion in this language to derive several of its properties. Moreover, the 2nd part of the course focuses on themes relating to the applications of the theory developed in the 1st part and adapts to the audience. We cover the theory of Large Deviations Principles and some of its applications to in the space of continuous functions, and we apply it to solve stochastic differential equations in a pathwise manner.

Prerequisites: SMSTC Probability 2-Stochastic Processes (or better)

Analysis of Diffusion Processes

The theory of diffusion processes has a very rich mathematical structure. One of the key features of such a theory is the interplay between probabilistic and analytic techniques. Analytic techniques are employed to give a macroscopic description of the dynamics, while probabilistic tools (stochastic analysis and stochastic processes) are used for the microscopic description.

The course will present the theory of time-homogeneous diffusion processes from the analysis standpoint.

Analysis of Diffusion Processes, cont'd

Part 1:

Markov Semigroups and their generators. Dual semigroup. Invariant and reversible measures

Ergodic Theory for Markov Semigroups: Strong Feller semigroups, Krylov-Bogoliubov Theorem, Doob's Theorem, Prokhorov's theorem

Diffusion processes: definition, relation with stochastic differential equations, Backward Kolmogorov and Fokker-Planck equation, Feynman-Kac

Reversible diffusions: spectral gap inequality, exponentially fast convergence to equilibrium

Over and under-dumped Langevin equation

Analysis of Diffusion Processes, cont'd

Part 2: hypoelliptic diffusions

Elliptic and hypoelliptic diffusions

The Hörmander condition (HC): this is a sufficient condition for hypoellipticity and will be presented by a probabilistic, analytic and geometric standpoint. In this respect, after introducing the HC (which stems from purely analytic considerations) we will discuss

Some basic notions of differential geometry: vector fields, integral curves and distributions (distributions in geometrical sense, not in probabilistic sense), orbits of a vector field

Geometric meaning of the HC: Chow's theorem and Sussman's orbit theorem

Probabilistic bearings of the Hörmander condition: existence of a density for the law of SDEs (The three above points may seem unrelated; as it turns out, they are very closely related indeed!)

Examples in statistical mechanics: heat bath models, second order Langevin equation

Analysis of Diffusion Processes, cont'd

Relation to other courses:

The background for this course is given by the Probability 1 SMSTC stream (and some lectures of the Probability 2 stream), where basic stochastic calculus and the basic theory of Markov Processes is covered. Moreover students with a background in analysis will find obvious relations with the theory of parabolic PDEs. There are strong links with the courses on dissipative PDEs as well.

Prerequisites: Basic probability theory, basic stochastic calculus (e.g. Ito formula), very basic SDE and PDE theory

What is ... Numerical Analysis?

A similar course "What is... PDEs?" currently runs informally at HW. The format is highly interactive, where students and upcoming seminar talks determine the content of the lectures. Interested students will give a 60-minute lecture on a topic of their choice, ideally a topic related to their research interests.

Aim: This course aims to give an introduction to standard techniques in the numerical analysis of partial differential equations, with a focus on the underlying analysis.

Prerequisites: a previous course in either PDE or their numerical analysis

Contents: We cover some essential basic and advanced topics in the numerical analysis of PDEs. After the course the student should know key ideas in a broad range of topics, as they are relevant in their research or in relevant numerical analysis talks.

What is ... Numerical Analysis?, cont'd

Basics I: Numerical methods, such as finite differences, finite elements, finite volume methods, boundary elements, time-stepping schemes

Basics II: Relevant topics in analysis, such as approximation properties of functions, Sobolev spaces and functional analysis

Finite element methods for elliptic problems: Conforming variational and mixed methods, error analysis, adaptive methods

Non-conforming and non-standard methods

Finite elements for the Stokes problem, analysis and stabilisation

Heat and wave equations: time-stepping schemes and their analysis

Fast solvers: review of numerical linear algebra, preconditioning, multigrid methods

Applications in computational mechanics, fluid dynamics or biology

Advanced PDE II: Hyperbolic PDEs

This course is dedicated to the study of hyperbolic PDEs in Sobolev spaces. We will mainly focus on nonlinear wave equations and symmetric hyperbolic systems.

The course will begin with some preliminary ideas including the method of characteristics, finite speed of propagation, and finite time blow-up.

Then, there will be a brief discussion of classical methods including the explicit solution formula of D'Alembert and Kirchhoff and the Cauchy-Kovalevskaya existence and uniqueness theorem.

The course will then turn to Sobolev space methods. These include energy estimates, Klainerman-Sobolev inequality, the vector-field method, and well-posedness of semilinear wave equations using Sobolev estimates.

Advanced PDE II: Hyperbolic PDEs, cont'd

In the final two weeks we will cover special topics at the instructor's discretion.

Possible special topics are small data global well-posedness of quasi-linear wave or Klein-Gordon equations (study started independently in works of Klainerman and Shatah from the 1980s), the global well-posedness of the defocusing energy-critical semilinear wave equation (work of Shatah-Struwe, 1994), or a survey of geometric hyperbolic PDE such as the Yang-Mills and wave maps equations.

Prerequisites: Students should be familiar with Banach and Hilbert spaces, dual spaces, and weak and strong convergence. The course "Advanced PDEs I" is suggested but not required.

Homogenisation I

Homogenization theory: multiscale modelling and analysis of physical and biological processes The aim of homogenization theory is to determine the macroscopic behaviour of a system comprising microscopic heterogeneities, e.g. transport processes in a porous medium, signalling processes in a cell tissue, deformations of composite materials. This means that the mathematical model defined in a heterogeneous medium is replaced by equations posed in a homogeneous one, which provide a good approximation of properties of the original microscopic system. In this course we will learn the main methods of homogenization theory, which are used to prove that solutions of microscopic problem, depending on a small parameter, converge to a solution of the corresponding macroscopic problem, as the small parameter (determined by the characteristic size of the microscopic structure) goes to zero.

Homogenisation I, cont'd

- Main methods of periodic homogenization: formal asymptotic expansion, two-scale convergence, unfolding operator
- derivation of compactness results for two-scale convergence and periodic unfolding operator
- multiscale modelling and analysis of fluid flow in porous media
- multiscale modelling and analysis of transport and reaction processes in perforated and partially perforated domains (i.e. signalling and transport processes in biological tissues, plant root growth)
- dual-porosity: modelling and multiscale analysis (transport and reaction processes in fractured media, in soil with porous particles, in cell tissues)
- multiscale analysis of equations of linear elasticity and viscoelasticity

Homogenisation II: Stochastic Problems

In this course, various concepts on stochastic homogenization theory are introduced with a view on multiscale modelling of multiphase systems. Stochastic homogenisation is a reliable and systematic theory for averaging partial differential equations defined on strongly heterogeneous media and domains with random characteristics. It has a wide range of applications from (reactive) transport in porous media, over to waves in heterogeneous media up to material science such as energy storage/conversion devices.It allows to rigorously derive effective macroscopic properties of strongly heterogeneous random media such as composite materials, the effective macroscopic formulation of microscopic systems, as well as the stable construction of multiscale computational schemes. We shall illustrate these features by considering various examples from continuum mechanics and physics of composite materials and porous media.

Homogenisation II: Stochastic Problems, cont'd

We begin with deriving effective stochastic differential equations (SDEs) based on the asymptotic two-scale expansion method. This provides a systematic tool to derive effective diffusion coefficients for heterogeneous materials. We will briefly discuss how to numerically solve SDEs and their effective/upscaled formulation. We then introduce the two-scale convergence methodology which is the basis for the stochastic two-scale convergence and the stochastic two-scale convergence in the mean. As before, we will discuss how to compute a numerical approximation of the resulting limit problems. The last topic of the course will be on general concepts of percolation theory and investigate similarities and differences to homogenization. Again, we will also give the basic ideas how to computationally study percolation problems. All these topics will be discussed based on examples and applications such as interacting particle systems under uncertainty/randomness, and if time allows we look also at the theory of fluctuations and correlations.

Homogenisation II: Stochastic Problems, cont'd

Recommended but optional prerequisites:

This course is a follow up on the course HOMOGENIZATION I, but students and researchers interested in stochastic averaging techniques will be able to follow it without having attended the first course.

The following experience is helpful but not required: Advanced PDEs 1 and basic knowledge in Probability Theory and stochastic differential equations and associated Kolmogorov equations. Useful are also basic knowledge about Measure and Integration Theory, and Functional Analysis. Basic knowledge about Galerkin/Finite Element approximations and Finite Difference methods.