SMSTC, Structure and Symmetry

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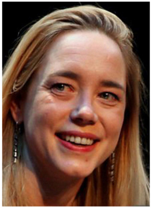
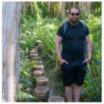


Colva Roney-Dougal



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Stream overview

Semester 1

- **Groups, Rings and Modules**
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- **Algebraic Topology**
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Semester 2

- **Algebras and Representation Theory**
  - Sira Gratz, Glasgow
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- **Manifolds**
  - Lorenzo Foscolo, Edinburgh
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Prerequisites

- **Groups, Rings and Modules**
  - Basic linear algebra and basic algebra concepts.
    - Definitions and examples of groups, rings and fields.
  - Basic notions of group theory.
    - Lagrange's theorem, normal subgroups and factor groups.
- **Algebraic Topology**
  - A course in metric spaces or topological spaces.
  - A course in group theory.
    - Group actions.
    - Finitely generated abelian groups.
- **Algebras and Representation Theory**
  - The notion of a module and related concepts.
  - Basics on Noetherian and Artinian modules.
  - Some commutative algebra.
- **Manifolds**
  - Standard calculus courses.
    - Green's theorem.
  - Basic courses in linear algebra.
    - Abstract vector space.

Groups, Rings, Modules and Representation Theory

- Groups.
  1. Simple groups, Jordan-Holder Theorem, (semi)direct products.
  2. Permutation representations and group actions.
  3. Sylow Thorems and applications.
  4. Abelian, soluble and nilpotent groups.
  5. Free groups and presentations.
- Commutative rings.
  1. Modules: introduction.
  2. Chain conditions and Hilbert's basis theorem.
  3. Fields and numbers.
  4. Affine algebraic geometry.
  5. Hilbert's Nullstellensatz.
- Noncommutative rings.
  - Finitely generated modules over principal ideal domains.
  - The Artin-Wedderburn Theorem.
- Representation Theory.
  - Representations and characters.
  - Orthogonality relations.
  - Induced representations.
  - Computing character tables.

Algebraic Topology and Manifolds

(1) Basic examples and constructions of topological spaces.
(2) Manifolds, basic homotopy theory and homotopy groups.
(3) Cofibrations, cell attachments and CW-complexes.
(4) Cellular approximation and relative homotopy groups.
(5) Fibre bundles, fibrations and the Hopf map.
(6) An introduction to homology.
(7) Homotopy invariance, exactness and excision.
(8) Computations and applications of homology.
(9) An introduction to cohomology.
(1) Implicit Function and Sard's Theorems, abstract manifolds.
(2) Tangent vectors and the tangent bundle, vector bundless.
(3) Vector fields and flows, Lie derivative, the Frobenius Theorem.
(4) Differential forms, Stokes' Theorem and Poincare duality.
(5) Riemannian metrics, connections, the Levi-Civita connection.
(6) Geodesics, the exponential map.
(7) Curvature and integrability, Riemannian curvature.
(8) Gauss Formula and the Theorema Egregium.
(9) Euler characteristic, the Gauss-Bonnet Theorem for surfaces.

Lüroth Problem

- Let $\mathbb{C}(x)$ be a field of rational function in 1 variable.
- Let $\mathbb{F}$ be a subfield in $\mathbb{C}(x)$ that contains $\mathbb{C}$.

Example

Let $\mathbb{F} = \mathbb{C}$.

Example

Let $\mathbb{F} = \mathbb{C}(x^2)$.

Example

Take any $f(x) \in \mathbb{C}(x)$. Let $\boxed{\mathbb{F} = \mathbb{C}(f(x))}$.

Question

Are there any other options for the subfield $\mathbb{F}$?

Theorem (Lüroth)

NO.

From fields to oriented surfaces

- The field $\mathbb{F}$ is generated by $f_1(x), \ldots, f_n(x)$ over $\mathbb{C}$.
- The functions $f_1(x), \ldots, f_n(x)$ are related by relations

$$\begin{cases} \mathbf{F}_1\big(f_1, \ldots, f_n\big) = 0, \\ \mathbf{F}_2\big(f_1, \ldots, f_n\big) = 0, \\ \ldots \\ \mathbf{F}_r\big(f_1, \ldots, f_n\big) = 0. \end{cases}$$

- This gives a subset Σ in $\mathbb{C}^n$ given by

$$\begin{cases} \mathbf{F}_1\big(x_1, \ldots, x_n\big) = 0, \\ \mathbf{F}_2\big(x_1, \ldots, x_n\big) = 0, \\ \ldots \\ \mathbf{F}_r\big(x_1, \ldots, x_n\big) = 0. \end{cases}$$

- One can choose generators of $\mathbb{F}$ such that Σ is *very good*.

Classification of compact oriented surfaces

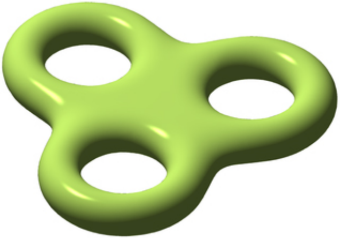
- The subset $\Sigma \subset \mathbb{C}^n$ is not compact.
- It can be *compactified* by squeezing $\mathbb{C}^n$ into $\mathbb{P}^n$.
- This gives a compact oriented surface **S** that contains Σ.
- Then $\mathbf{S} \setminus \Sigma$ consists of finitely many points.
- And $\mathbb{F}$ is a field of rational function of the variety **S**.

Theorem

The surface **S** *is diffeomorphic to a sphere with* **g** *handles attached.*

Example

If Σ is given by $x^3y + y^3 + x = 0$ in $\mathbb{C}^2$, then **S** looks like



Importance of being a sphere

Recall that $\mathbb{F}$ is a subfield in $\mathbb{C}(x)$ that contains $\mathbb{C}$.

Lemma

$\mathbb{F} = \mathbb{C}(f(x))$ *for some* $f(x) \in \mathbb{C}(x) \iff \mathbf{g} = 0$.

Proof.

- $\Longrightarrow$ is clear ($\Sigma = \mathbb{C}$ and $\mathbf{S} = \mathbb{P}^1$).
- $\Longleftarrow$ follows from Riemann–Roch theorem.

□

Since $\mathbb{F}$ is contained in $\mathbb{C}(x)$, we obtain a map

$$\boxed{\mathbb{C}^1 \longrightarrow \mathbf{S}}$$

which is *almost* surjective. It gives a surjective map

$$\boxed{S^2 \longrightarrow \mathbf{S}.}$$

If it is one-to-one, then we are done.

Euler characteristic

- We have constructed a surjective map $\phi\colon S^2 \to \mathbf{S}$.
- We want to show that $\mathbf{g} = 0$.

Let d be the number of points in $\phi^{-1}(P)$ for general $P \in \mathbf{S}$. Then

$$\boxed{\left|\phi^{-1}(P)\right| \leqslant d}$$

for every $P \in \mathbf{S}$. Let Δ be a finite subset in $\mathbf{S}$ such that

$$\boxed{\left|\phi^{-1}(P)\right| < d}$$

for every $P \in \Delta$. Let $\nabla = \phi^{-1}(\Delta)$. Then

$$2 = \chi\big(S^2\big) = \chi\big(S^2 \setminus \nabla\big) + \chi\big(\nabla\big) = \chi\big(S^2 \setminus \nabla\big) + \big|\nabla\big| =$$
$$= d\chi\big(\mathbf{S} \setminus \Delta\big) + \big|\nabla\big| \leqslant d\chi\big(\mathbf{S} \setminus \Delta\big) + d\big|\Delta\big| = d\chi\big(\mathbf{S}\big) = d\big(2 - 2\mathbf{g}\big),$$

which implies that $\mathbf{g} = 0$.

Artin–Mumford counterexample

Let $F_2 = x^2 + y^2 + z^2 + t^2 + (x + y + z + t)^2$ and

$$F_4 = x^4 + y^4 + z^4 + t^4 + \big(x + y + z + t\big)^4.$$

Let $\mathbb{F}$ be the field of fractions of the ring

$$\mathbb{C}\big[x, y, z, w\big]\Big/\Big\langle w^2 - F_4(x, y, z, 1) + \frac{1}{2}F_2^2(x, y, z, 1)\Big\rangle.$$

Let X be the hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$ that is given by

$$w^2 = F_4(x, y, z, t) - \frac{1}{2}F_2^2(x, y, z, t).$$

Then $\mathbb{F}$ is a field of rational function of the variety X.

- The variety X has 10 ordinary singular points.
- Let $\psi\colon V \to X$ be the blow up of these points.
- Then $\mathrm{Br}^{nr}(\mathbb{F}) \cong H^3(V, \mathbb{Z}) \cong \mathbb{Z}_2$.

This implies that $\mathbb{F} \ncong \mathbb{C}(x, y, z)$. But $\mathbb{F} \subset \mathbb{C}(x, y, z)$ (easy).