Pure Analysis Stream
Measure and Integration (Semester I)
Functional Analysis (Semester II)

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5 Oct, 2017

Outline

1. Content of the two courses
   - Measure and Integration
   - Functional Analysis
2. Prerequisites
3. Method of assessment

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Integration as a tool

- Integration as a tool
  - Geometry, Representation Theory, Probability, Applied Analysis, etc...
- Geometry – Gauss Bonnet Theorem

$$\int_M K \mathrm{d}A \quad = \quad 2\pi \, \chi(M)$$

- Applied analysis – solving differential equations
- Representation theory: average over compact group by integrating over the group.

Riemann Integral

- Often the definition of the integral introduced in first undergraduate analysis course treating differentiation and integration.
- Good enough for many purposes
- Has bad limiting properties: conditions under which

$$\lim_n \int f_n = \int (\lim_n f_n)$$

quite restricted.
- Particular problem when working with Fourier series:

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi inx}, \quad a_n = \int_0^1 f(x) e^{-2\pi inx}\, dx$$

- Need a more powerful integral: The Lebesgue theory.

Abstract Measure and Integration

- Abstract measure spaces and Lebesgue integral
- Setting for Probability theory
  - A *measurable* set E corresponds to an **event**
  - A *measurable* function corresponds to a random variable
- Two lectures (foundations and complex measures)

This abstract framework will unify:

- Riemann and Lebesgue integral on $\mathbb{R}$: $\int_{\mathbb{R}} f(x)\,dx$
- Infinite series: $\sum_{n=1}^{\infty} a_n$
- Elements of $C(X)^*$ (the continuous dual space of $C(X)$):

$$\Lambda \in C(X)^* \implies \Lambda(f) = \int_X f(x) d\mu(x)$$

Constructing measures

- Construction of Lebesgue measure on real line
- Carathéodory construction.
  - using outer measures
  - fractal sets and Hausdorff dimensions
- Product measures: $X = Y \times Z$ (e.g. $\mathbb{R}^2$
  - Fubini's theorem: when is it legitimate to change the order in a double integral.
- Radon measures – dual space of $C(X)$
- Total of 4 lectures (actually we'll construct Lebesgue measure on $\mathbb{R}$ before developing the abstract framework).

Further topics

- $L^p(X)$ spaces – $\int_X |f|^p < \infty$. Crucial to harmonic analysis.
- Differentiation
  - $F(x) = \int_a^x f(x)dx \implies F'(x) = f(x)$
  - Maximal functions
- End with brief discussion of Fourier series and overview of where research goes from here.

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Functional Analysis

- Analysis on infinite dimensional vector spaces (functional analysis = linear analysis).
- Many key examples are spaces of functions:
  - the L^p spaces from first semester.
  - Sobolev spaces; essential for differential equations.
- Provides tools used right across mathematics:
  - Baum-Connes conjecture (using language of operator algebras) implies Novikov conjecture in topology, Kaplansky conjecture in algebra.
  - Most general statements known about Novikov and Kaplansky factor through functional analysis.

Basics

- Banach and Hilbert spaces

  $L^p(X)$ spaces, $C(X)$, $C(X)^*$, etc ...
  Setting for Fourier series

- Linear operators and linear functionals
- Fundamental theorems: in the 'Scottish book' from cafe in Lwów.

  Baire Category, Open Mapping, Uniform Boundedness Principle, etc...

- Three lectures

Dual spaces and weak topologies

- Weak and weak* topologies
  - Banach-Alaoglu (weak* compactness of unit ball)
- Convexity in infinite dimensional setting.
  - Krein-Milman Theorem (very useful convexity result)
  - Use to produce Haar measure on compact groups.
- Two lectures

Spectral Theory

- Compact operators and their spectra
- General spectral theory

  Banach algebras
  Spectral theorem for self-adjoint operators
- Commutative C^*-algebras – Gelfand's theory
- Four lectures; again aiming to finish with overview lecture pointing towards current research in this direction.

Prerequisites for both courses

Formal prerequistes for both courses

- Undergraduate real analysis
  - Sequences, series, pointwise and uniform convergence.
  - Continuous and differentiable functions, basic properties.
- Metric space topology (at least in $\mathbb{R}^d$)
  - continuity of functions, open, closed and compact sets
- Countablilty
  - A set S is **countable** if $S = \{s_1, s_2, s_3, \ldots\}$.
  - The set of reals $\mathbb{R}$ is **uncountable**!

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For: Measure and Integration

- Will start with full account Riemann integral, so self contained.
- If you've not seen the Riemann integral (or a version of the Lebesgue integral on $\mathbb{R}$) before, you'll need to be motivated.

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For: Functional Analysis

- If seen measure theory and some Fourier analysis before ideal!
- Without measure theory, should be able to get a lot out of the course, but need to ignore some of the examples.

Assessment for both courses

Ongoing feedback

- Exercises associated to each lecture.
- Not part of formal assessment; for discussion in tutorials.
- Some lectures (certainly including me) will ask for groups to present some of these in lectures.

Formal assessment for each course

- Two assignments for each course; normally consisting of three questions per assessment.
- In particular, there'll be no measure theory assumed in the functional analysis assessment