SMSTC: Probability Stream

Fraser Daly

Heriot–Watt University

October 2017

Outline

- Probability
- Course outlines and teaching team
- Prerequisites
- Assessment
- Feedback

Probability

"Chance, too, which seems to rush along with slack reins,
is bridled and governed by law."

– Boethius (ca. 480–505),
The Consolation of Philosophy

Probability

- mathematical modelling of uncertainty: random events, random processes evolving in time.
- foundations in measure theory, analysis, functional analysis, combinatorics.
- strongly driven by physical intuition and ideas of information evolving in time.

Foundations of Probability (Semester 1)

- **Fundamentals**: probability spaces, σ-algebras, probability measures, conditioning and independence
  David Siska (Edinburgh)
- **Random variables** and their distributions, important special distributions (binomial, Poisson, geometric, normal, exponential etc.)
  David Siska (Edinburgh) and Fraser Daly (Heriot-Watt)
- **Convergence** and **limit theorems**
  Sergey Foss (Heriot-Watt)
- **Conditional expectation** and **martingales**
  Michela Ottobre (Heriot-Watt)
- **Renewal theory**
  Fraser Daly (Heriot-Watt)

Stochastic Processes (Semester 2)

- **Markov chains** and **processes**, **Poisson processes**
  Burak Buke (Edinburgh)
- **Applications**, including connections to statistics and graph theory
  James Cruise (Heriot-Watt)
- **Brownian motion** and **stochastic calculus**
  István Gyöngy (Edinburgh)

An example

A gambler starts with £X_0. At turn $n = 1, 2, \ldots$, he stakes £S_n, and

- gains £S_n with probability $p > 1/2$, or
- loses £S_n with probability $1 - p$.

We let £X_n be his total wealth after turn n, and assume (reasonably!) that $0 \leq S_n \leq X_{n-1}$.

How can the gambler maximize his long-term gain?

Calculations using *conditional expectation* show that $E(X_n)$, the gambler's average wealth after turn n, is maximised by choosing $S_n = X_{n-1}$. But, this is not a viable long-term strategy (what happens the first time you lose?)...

An example

If we instead try to maximise $E\log(X_n)$, we can show that this is achieved using the strategy $S_n = (2p-1)X_{n-1}$.

One way to do this is to show that a certain linear shift of $\log(X_n)$ is a *martingale* in this case, and a *supermartingale* in all others.

We can also check, using the *law of large numbers*, that if

- our gambler uses this strategy, and has £X_n after tun n, and
- another gambler uses the strategy $\widetilde{S}_n = \lambda\widetilde{X}_{n-1}$ (where $\lambda < 1$ and $\lambda \neq 2p-1$), and has £$\widetilde{X}_n$ after turn n

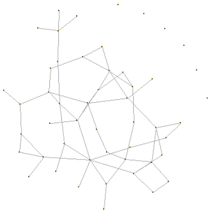
then $X_n/\widetilde{X}_n$ grown exponentially for large n, with probability 1. Hence, the choice $\lambda = 2p-1$ is a better choice than any other.

Erdős–Rényi random graph

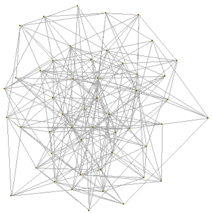
Suppose we have n vertices/nodes.

Each pair of vertices is joined by an edge/link with probability p, independently of all other pairs of vertices.

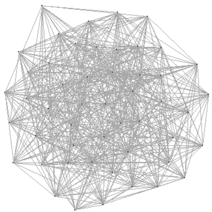
This is the Erdős–Rényi random graph $G(n, p)$. It can be used to model a 'typical' (or 'unstructured' or 'random') communication (or power, or distribution, or ...) network, for example.



$p = 0.05$



$p = 0.2$



$p = 0.5$

How long is the longest path?

Let $p = c/n$ (with $0 < c < \log n - 3 \log\log n$). Then $G(n, p)$ contains a path of length at least

$$\left(1 - \frac{4 \log 2}{c}\right) n \, ,$$

with probability 1, for large enough n.

This is is proved by analysing an algorithm which explicitly constructs such a path, and exploiting the *Markovian* structure present in the algorithm.

Colouring the complete graph

Let K_n be the complete graph, with n vertices and an edge between each pair of vertices. Suppose we colour each edge of K_n either red or blue.

There is a colouring of K_n which contains at most $\binom{n}{a}2^{1-\binom{a}{2}}$ monochromatic copies of the complete graph K_a.

We can prove this by

- Randomly colouring K_n (each edge is red with probability $1/2$, or blue otherwise, independently of the other edges);
- Calculating that the average number of monochromatic copies of K_a is $\binom{n}{a}2^{1-\binom{a}{2}}$; and
- Concluding that there must exist a colouring with at most this many monochromatic copies of K_a.

Prerequisites

- Elements of mathematical analysis, linear algebra and combinatorics at undergraduate level.
- For Stochastic Processes, in addition: Probability theory, either at undergraduate level or from Foundations of Probability.
- the ability to think both rigorously and intuitively!

Some suggestions for further/background reading are on the handout.

Assessment

"I can see looming ahead one of those terrible exercises in probability where six men have white hats and six men have black hats and you have to work it out by mathematics how likely it is that the hats will get mixed up and in what proportion. If you start thinking about things like that, you would go round the bend. Let me assure you of that!"

– Agatha Christie, *The Mirror Crack'd from Side to Side*

Assessment

Each module is assessed by two written assignments.

Provisional deadlines on:

- Foundations of Probability: 21 November 2017 and 9 January 2018.
- Stochastic Processes: 20 February 2018 and 3 April 2018.

Assignments will be available at least two weeks before the deadline.

Solutions for (at least) one assignment from each module should be prepared using $\LaTeX$.

Feedback

- is a two-way process.
- if you have any questions/concerns, get in touch with me (f.daly@hw.ac.uk, 0131 451 3212) or another member of the teaching team.
- please don't wait for the end of the course!