SMSTC Course: Homotopy Theory

Irakli Patchkoria

University of Aberdeen, Department of Mathematics

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- Homotopy theory: Study spaces and continuous maps up to homotopy, that is up to continuous deformations.
- Main goal: Study continuous maps up to homotopy between spheres $S^n \longrightarrow S^m$.
- Main object of study: $\pi_n S^m = [S^n, S^m]$, (pointed) homotopy classes of maps between the spheres.

Two semesters

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- Further literature is given in the course description.

Prerequisites

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- Basic algebraic topology: fundamental group, covering spaces, singular homology and cohomology (cup product)
- Basic commutative, linear algebra, homological algebra

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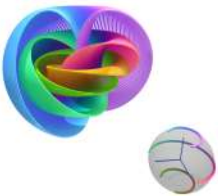
- Stably $\pi_1 \mathbb{S} \cong \pi_4 S^3 \cong \pi_5 S^4 \cong \pi_6 S^5 \cong \cdots \cong \mathbb{Z}/2$. Generated by higher and higher suspensions of η.

The Hopf map $S^3 \longrightarrow S^2$



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- $\pi_k \mathbb{S} \cong \pi_{n+k} S^n$ is finite for $k \geq 1$ and $n >> 0$ $(k+2 \leq n)$.
- E.g. $\pi_1 \mathbb{S} \cong \mathbb{Z}/2$.

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- Use homological algebra: The *Adams spectral sequence* which is a procedure like a chess board:

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s} \mathbb{S}^{\wedge}_2,$$

where $\mathcal{A}_*$ is the dual Steenrod algebra.

Visualisation of the Adams spectral sequence for computing $\pi_k\mathbb{S}$

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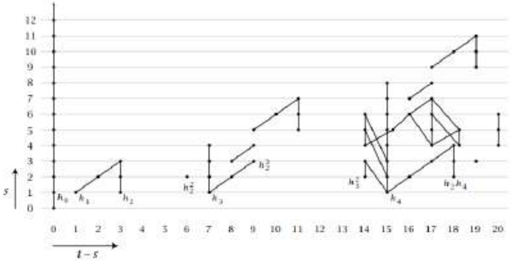

1	2	3	4	5	6	7
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$

Lecturers

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- First semester: Rachael Boyd (Glasgow), Livio Ferretti (Glasgow), Mark Grant (Aberdeen), Irakli Patchkoria (Aberdeen), Mark Powell (Glasgow)
- Second Semester: Ran Levi (Aberdeen), Irakli Patchkoria (Aberdeen), Mark Powell (Glasgow), ...

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- Each lecturer will set and grade one homework.

Thanks!

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