Applications of Mathematics

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30 September 2025

Overview of the theme

The theme can be divided into two broad categories:

- the formulation and analysis of *mathematical models*
- *methods or tools* needed to analyse and simulate them

Two modules per category; self-contained and independent.

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Two major areas of modelling:

- **continuum mechanics**
- **mathematical biology**

Not comprehensive, but illustrative of a range of approaches.

Two complementary sets of methods:

- **asymptotic and analytic**
- **numerical**

Again not comprehensive, but illustrative. In both cases the aim is to develop *approximate* methods in a systematic, quantitative way.

Continuum Mechanics (semester 1)

❍ Continuum mechanics describes how deformable media behave:

- Fluids (liquid/gas)
- Solids (elastic/plastic)

Continuum hypothesis + Newton II:

$\Rightarrow$ $O(10^{24})$ molecules: can't solve $F = ma$ for each

$\Rightarrow$ Treat medium as a continuum of parcels, each small on scale of motion, each containing large number of molecules

$\Rightarrow$ Then $\mathbf{u}$, p, ρ, etc, considered as functions of $(\mathbf{x}, t)$

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- Huge range of applications (nano-technology to astrophysics)

Continuum Mechanics (semester 1)

Topics:

- **Continuum mechanics:** construction of dynamical models of deformable media
- **Fluid Dynamics:** lubrication theory, aerofoils, hydrodynamic stability
- **Non-Newtonian fluids:** fluid viscosity depends on internal stresses

Mathematical Biology and Physiology (semester 2)

- Mathematical modelling in the life sciences.

- Modelling of just about any aspect of biological systems:
  - Circulation (e.g., blood flow in arteries)
  - Patterns (e.g., rashes)
  - Populations dynamics
  - Cell dynamics (dynamics at cell scale)
  - Diseases/treatments (epidemiology)

Mathematical Biology and Physiology (semester 2)

Topics:

- **Bacterial resistance:** antibiotics
- **Mathematical Physiology:** microscale/macroscale & homogenisation
- **Population Modelling:** epidemiology, evolution, pathogen-host interactions, age-structured models
- **Mathematical Oncology:** cancer modelling

Problems frequently contain a "small parameter" $\epsilon \ll 1$.

- Can change the character of the problem

$$i\epsilon\partial_t u + \epsilon^2 \Delta u + Vu = 0.$$

Asymptotic and Analytical Methods (semester 1)

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- Can change the character of the problem

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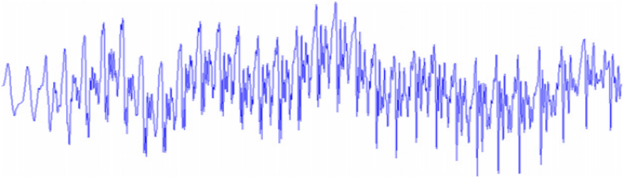
Some examples:

- In PDEs (e.g. Navier–Stokes, Schrödinger), small terms may give rise to boundary layers or caustics.



- In forced oscillators, small forcings may lead to resonance.

These lectures will provide a toolkit for tackling such problems.

EXAMPLE: Asymptotic expansion of the error function:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \qquad (< 1).$$

Seek an expansion to evaluate $\mathrm{erf}(x)$ for large x.

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❍ First attempt: Taylor expand e^{-t^2} and integrate term-by-term:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{1}{5}\frac{x^5}{2!} - \ldots + \frac{(-1)^n}{2n+1}\frac{x^{2n+1}}{n!} + \ldots \right]$$

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- **Convergent:** gets arbitrarily close to $\operatorname{erf}(x)$ for any x
- $x = 4$: needs > 40 terms for a reasonable approximation
- $x = 4$: first 16 terms sum to 10^5

❍ Alternative approach: Write

$$\operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

and integrate by parts n times:

$$\int_x^\infty e^{-t^2} dt = \int_x^\infty \frac{te^{-t^2}}{t} dt$$
$$= e^{-x^2} \left[\frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5} + \ldots + \frac{(-1)^{n-1}}{2^n} \frac{1.3.5\ldots(2n-3)}{x^{2n-1}} \right] + R_n$$

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- $x = 4$: first 2 terms give erf(4) to within 10^{-7}

Asymptotic and Analytical Methods (semester 1)

Topics:

- **Multiple scales:** modulation and resonance
- **Matched Asymptotics:** boundary layers
- **WKB Theory:** rapidly oscillating solutions, ray theory
- **Approximation of Integrals:** Watson's lemma, steepest descent
- **Intermediate Asymptotics:** self-similarity of solutions
- **Resummation:** techniques for series

Numerical Methods (semester 2)

- **Ways to solve mathematical models numerically.**
- Often there is a *compromise* between ease of implementation and efficiency.
- Or between speed (computational cost) and accuracy.

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Topics:

- **Ordinary DEs:** Explicit, implicit and multistep methods
- **Stochastic DEs:** Brownian Motion, Stochastic Integrals, Stochastic DEs and simulations
- **Partial DEs:** Finite Difference and Finite Element methods
- **Linear Algebra:** Numerical Solution of Linear Systems and Eigenvalue problems.

Numerical Methods (semester 2)

❍ Numerical methods for PDEs

Many models in applied mathematics are based on PDEs.

- Many pieces of software claim to be able to solve PDEs.
- Pretty pictures are not always accurate!!!
- It is good to know what are some standard things that can go wrong.

Numerical Methods (semester 2)

❍ Numerical methods for PDEs

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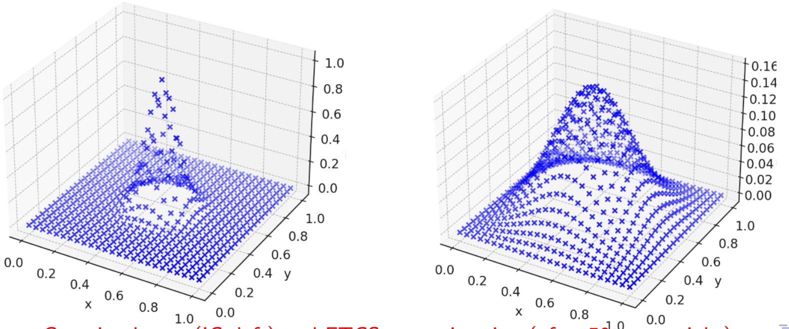
These lectures will look at some of the basic questions:

- How can we discretise PDEs to give systems of algebraic equations?
- How can we solve the resulting systems efficiently?
- How do we prove these methods have desirable properties, like
  - Consistency (representing the desired PDE)
  - Stability (not being hypersensitive to data)
  - Convergence (error decreasing as resolution increases)

❍ EXAMPLE: FTCS method for the heat equation: $\partial_t u = \Delta u$ (homogeneous Dirichlet BCs):

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t} = \frac{U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n}}{(\Delta x)^2} + \frac{U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}}{(\Delta y)^2}.$$

- If $\Delta x = \Delta y$, choose $\Delta t < \frac{(\Delta x)^2}{2}$ to ensure *stability* and hence *convergence*.

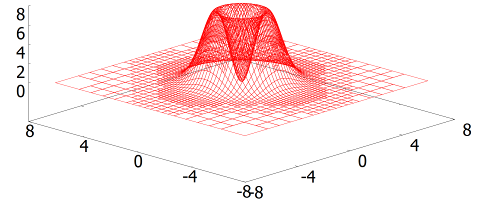


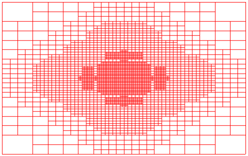
Gaussian bump (IC; left) and FTCS approximation (after 50 steps; right)

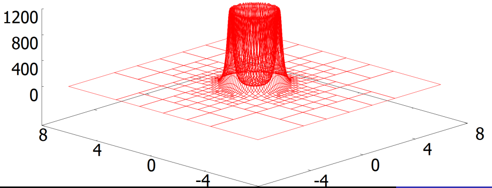
Numerical Methods (semester 2): Finite Elements: More Advance Techniques!

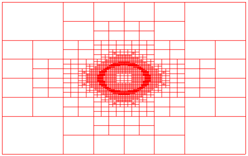
Semilinear parabolic equations: Regional blowup

- $\partial_t u - \Delta u = u^2$ in $\Omega \times (0, T]$, $\Omega = (-8, 8) \times (-8, 8)$
- $u_0(x, y) = 10(x^2 + y^2)\mathrm{e}^{-0.5(x^2+y^2)} \rightsquigarrow$ Blowup set: circle centred on $(0, 0)$



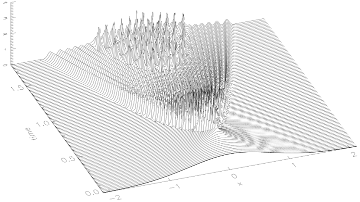




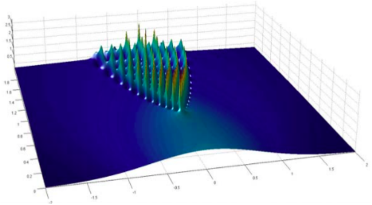


- Complicated geometries
- Sharply localised features

- *Semiclassical behaviour* of NLS



Cai, McLaughlin & McLauglin, 2002



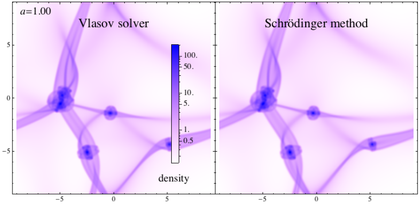
Bertola & Tovbis, 2013

NLS dispersive breaking

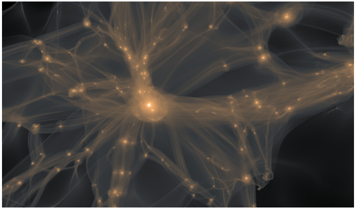
Numerical Methods (semester 2):
Finite Elements: More Advance Techniques!

- Complicated geometries
- Sharply localised features

- Cosmology: *Galaxy formation*



Kopp, Vattis & Scordis, 2017



From the page of Dr. R. Kaehler

2d & 3d simulations for galaxy formation

Prior knowledge, delivery and assessment

Assumed knowledge of standard, core undergraduate material:

- Calculus, ODEs, PDEs
- Linear Algebra
- Complex Variables

More details on the module pages.

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Assessment: two assignments per module

- Mix of analytic and computational work
- Normally at least two weeks to complete each