Travelling Waves in Parabolic Equations

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Setup

Travelling Waves in Parabolic Equations is a course in **Applied Analysis:** we use methods of Mathematical Analysis to get to understand mathematical objects appearing in the natural sciences (physics, chemistry, biology).

Parabolic Equations?

The simplest example of the object we are interested in is a **semilinear scalar reaction-diffusion equation** in which the unknown $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbb{R}, \quad t > 0, \tag{1}$$

with (for example) $f(u) = u(1-u)$ plus some suitable initial condition $u(x,0) = u_0(x)$.

Very often $u(x,t)$ is a **density** of some sort or another, and then we also require $u(x,t) \geq 0$ for all x and t.

This example can be made more complex in a wide variety of ways:

- add an **advection** term: e.g. $u\dfrac{\partial u}{\partial x}$;
- change the **diffusion mechanism**: e.g. $\dfrac{\partial}{\partial x}\left(u\dfrac{\partial u}{\partial x}\right)$;
- add terms to account for nonlocal interactions;
- consider systems of equations instead of a scalar equation.

The literature is vast because . . .

. . . such equations and their travelling waves have many applications, *inter alia*, in epidemiology, chemical reaction theory, combustion and detonation theory, population dynamics and population genetics, liquid crystals, gas dynamics, etc., etc..

Travelling Waves?

A **travelling wave** solution of an equation such as (1) is a solution $u(x,t) = U(z)$, where $z = x - ct$ is the *travelling wave variable* and c is the *wave speed*. There are all kinds of travelling waves one can expect: periodic waves, solitons, fronts. As an example of a (monotone) front, consider

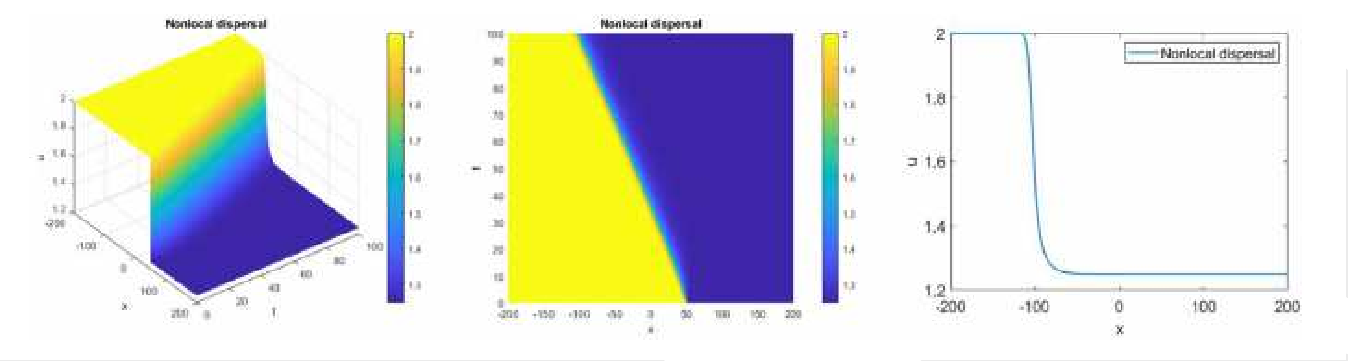

Figure 1: A propagating monotone front from B.-E. Jiang, F.-Y. Yang and W.-Y. Tang, ZAMP **74**(2023), 201.

Study of travelling waves effectively reduces a PDE problem to an ODE problem and so in particular makes methods and concepts of Dynamical Systems theory applicable.

Analysis of travelling waves of equations such as (1) started in the 1930s with the fundamental work of A. Kolmogorov (1903–1987) and has attracted important mathematicians and physicists such as N. Goldenfeld, **K.-P. Hadeler** (1936–2017), L. Nirenberg (1925–2020), L. Ryzhik, W. van Saarlos, H. Weinberger (1928–2017) and many others.

Topics

- Existence (e.g. of monotone fronts);
- Stability and questions of convergence;
- Existence of explicit solutions;
- **Minimality**;
- Other aspects (?): voting models, renormalisation group methods.

Explanation of **minimality**: Very often there is a semi-infinite interval $c \in [c_0, \infty)$ of possible speeds. The travelling wave with the minimal speed c_0 is what is physically observed. Much effort has been expended at finding or at least estimating this minimal speed (which tells you, for example, how fast an LCD device can switch a pixel).

We will also discuss the rather mysterious connection between minimality and the existence of explicit solutions.

Prerequisites and Administration

Prerequisites: A good knowledge of ODE techniques and concepts is recommended (the 1st half of the Dynamical Systems and Conservation Laws course is excellent preparation). Some Functional Analytic concepts will be used, but these aspects will be explained from scratch.

Administration: 20 hours of lectures; two equally weighed assignments, one in the middle of the course and one at the end.