Introduction to the Geometry of Elliptic Curves

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What is an elliptic curve?

An **elliptic curve** is a smooth projective algebraic curve of genus 1, equipped with a distinguished point O (the *origin*). m,ll

Example: smooth plane cubic (Weierstrass Model)

It can be written as

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{C},$$

with discriminant

$$\Delta = -16(4a^3 + 27b^2) \neq 0$$

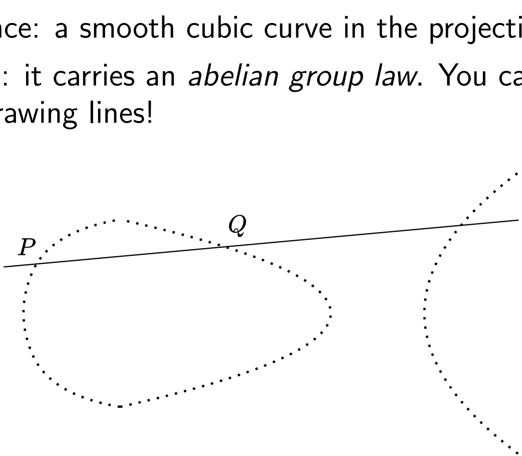
to ensure smoothness.

- The point at infinity $[0 : 1 : 0]$ is a distinguished point O in this case.

What is an elliptic curve?

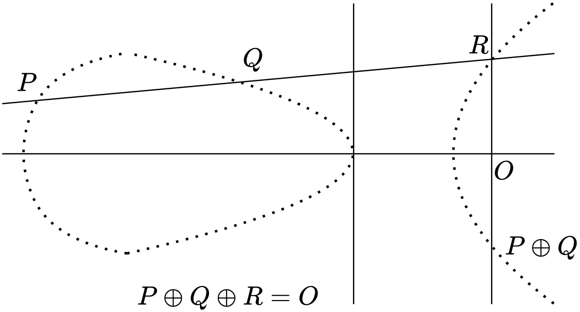
At first glance: a smooth cubic curve in the projective plane.

Secret twist: it carries an *abelian group law*. You can *add points* by drawing lines!



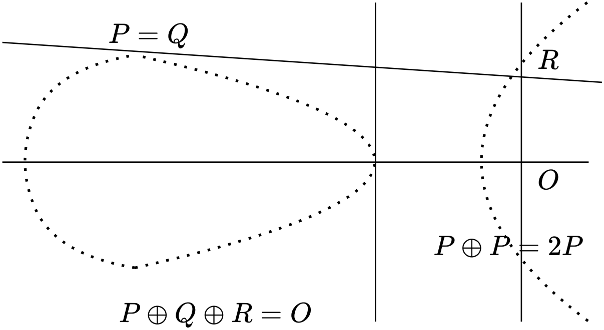
How to add points

- Draw the line through P and Q; it meets the curve again at the third point R.
- Reflect R to get $P \oplus Q$.



How to add points

- Tangent at P gives $P \oplus P = 2P$ (same recipe).



Moral: geometry $\Rightarrow$ algebra.

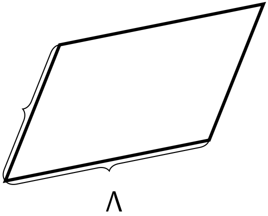
Fun fact: every complex elliptic curve is a donut

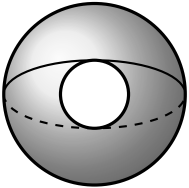
Over $\mathbb{C}$, every elliptic curve E is isomorphic to a complex torus:

$$E \cong \mathbb{C}/\Lambda,$$

where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ is a lattice with ω_1, ω_2 linearly independent over $\mathbb{R}$.

- Quotienting $\mathbb{C}$ by Λ identifies points differing by lattice vectors.
- Geometrically: a **donut-shaped surface (torus)**.



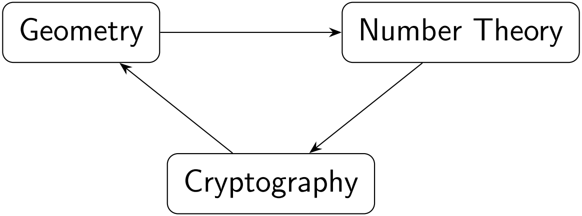


Why should we care?

Geometry
Curves, divisors, line bundles, Riemann–Roch.

Number Theory
Diophantine equations, ranks.

Cryptography
Public-key protocols on the group of points.



Course Aims

- Introduce graduate students to the geometry of algebraic curves.
- Understand elliptic curves geometrically (over $\mathbb{C}$).
- Cover group law and isomorphisms.
- Develop key tools: divisors, line bundles, morphisms, Riemann–Roch.
- Prepare students for further study in algebraic geometry and related areas.
- Practice research communication via a short talk.

Prerequisites

Students are expected to have prior exposure to:

- Abstract algebra (groups, rings, fields)
- Basic algebraic geometry (affine and projective varieties, morphisms)

Overview

1. Curves and projective geometry
2. Singularities and smooth curves
3. Genus and Riemann–Roch
4. Definition of elliptic curves
5. Group law
6. Isomorphisms and j-invariant
7. Complex tori and $\wp$-function
8. Line bundles and morphisms
9. Applications of Riemann–Roch
10. Elliptic curves over other fields (optional)

Assessment

Final Presentation (100% of grade)

- 15-20 minute individual talk
- Topic chosen in consultation with the lecturer
- Emphasis on conceptual clarity and exposition

Main Reference

- J.H. Silverman, *The Arithmetic of Elliptic Curves*

Closing

- Elliptic curves: a central object in modern mathematics
- Skills gained:
  - Geometric intuition
  - Algebraic techniques
  - Expository skills
- Questions and discussion welcome!