$\mathscr{D}$-modules and the Riemann-Hilbert correspondence

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VIA VERITAS VITA

Outline

- What are $\mathscr{D}$-modules?
- Why were they invented?
- Why should you (do I) care?
- Questions

What are $\mathscr{D}$-modules?

Differential operators

$$D_1 = x_2^2 \partial_1^3 - 4(x_1 - x_2)^2 \partial_2, \quad D_2 = 5x_1^2 \partial_1 + (x_1^3 - 5),$$

act on functions, where

$$\partial_i = \frac{\partial}{\partial x_i}.$$

So we can compose and add them:

$$D_1 \circ D_2, \quad D_1 + D_2$$

to get new differential operators.

What are $\mathscr{D}$-modules?

- Set $\mathscr{D}(\mathbb{C}^n)$ of all differential operators is a ring.
- $D_1 \circ D_2 \neq D_2 \circ D_1$ so $\mathscr{D}(\mathbb{C}^n)$ is (very!) non-commutative.
- $\mathscr{D}$-module just means a module over $\mathscr{D}(\mathbb{C}^n)$.

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- Consider the system $P_1(f) = P_2(f) = P_3(f) = 0$, where

$$P_1 = x_1 + \partial_2, \quad P_2 = x_2^2\partial_1 - 4, \quad P_3 = x_1^2\partial_1 - x_2\partial_2^2.$$

- What is the dimension of the space S of solutions?

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- What is the dimension of the space S of solutions?
- If $M = \mathscr{D}(\mathbb{C}^2)/(P_1, P_2, P_3)$ then $\mathrm{Hom}_{\mathscr{D}}(M, \mathcal{O}) \cong S$.
- Pass to geometry: if

$$\dim V(x_1 + y_2, x_2^2 y_1, x_1^2 y_1 - x_2 y_2^2) = 1.$$

- Gabber's Theorem + symplectic geometry implies $M = 0$.

Why should you (do I) care?

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- Essential tool to understand rep theory of Lie algebras, Hecke algebras and DAHAs, quantizations...
- Many applications in algebraic geometry.

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Riemann-Hilbert correspondence:

$$\mathscr{D}\text{-modules} \to \text{Analysis} \to \text{Topology}$$

The solution functor $M \mapsto \mathrm{Hom}_{\mathscr{D}}(M, \mathcal{O})$ gives equivalence

$$D^b_{\mathrm{rh}}(\mathscr{D}_X) \cong D^b_{\mathrm{cs}}(\mathbb{C}_X).$$

Provides deep link between representation theory and topology.

Sheaves

Do I need to know about sheaves?

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Yes and no.

- Sheaves are essential for functorality and for the Riemann-Hilbert correspondence.
- Will provide a "cheat sheet" with basic facts we will use.
- Can be treated as a black box.

Derived categories

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Yes and no.

- Derived categories are essential for functorality and for the Riemann-Hilbert correspondence.
- Can be treated as a black box.
- Proofs of the key results will not use derived categories.