Resurgence

In Geometry and Physics

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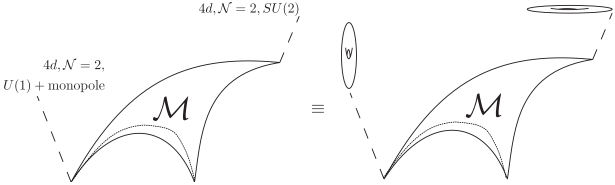
Wonderful things happen at the intersection of mathematics and physics

- Packaging the data of physical theories in geometric terms has been of enormous benefit for mathematics and physics
- Parameter spaces become geometric moduli spaces $\mathcal{M}$, the change is captured by variation and wall-crossing problems



Seiberg and Witten '94

Mathematics elucidates deeper physical structures

Physical perturbation theory?

- Perturbative results which are challenging to interpret, $Z(\tau, a) \sim \sum_{n=0}^{\infty} \tau^n Z_n(a)$ asymptotic series.
- Non-perturbative effects? $Z(\tau, a) + \exp(-1/\tau) \ldots ?$

Dualities?

- Theory A at coupling τ, gauge group $G \sim$ Theory B at coupling $1/\tau$, gauge group ${}^L G$

Physics reveals overarching principles in pure mathematics

Is modern mathematics organized in terms of quantum physics?

- Mathematical invariants are organized in terms of physical partition functions.
- Generating functions of geometric invariants are naturally associated to quantization problems, e. g. the Jacobi theta function $\theta(z, \tau)$

Dualities reveal connections within mathematics

- Mirror symmetry connects complex and symplectic geometry, uses representation theory and connect to number theory
- Geometric Langlands duality can be understood as $\tau \to 1/\tau$, $S-$duality

Resurgence gives a systematic way to handle asymptotic series

Suppose

$$\phi(z) = \sum_{n\geq 1} a_n\, z^n\,,$$

is an asymptotic series around 0 with $a_n \sim n!$

Its Borel transform is then given by the series:

$$G(\xi) = \sum_{n\geq 1} \frac{a_n}{(n-1)!} \xi^{n-1}\,.$$

The Laplace transform gives the Borel sum:

$$\phi_\rho(z) = \int_\rho e^{-\xi/z} G(\xi) d\xi\,.$$

Resurgence gives a systematic way to handle asymptotic series, simple example

Suppose

$$\phi(z) = \sum_{n \geq 1} (n-1)!\, z^n \,,$$

is an asymptotic series around 0 with $a_n \sim n!$
Its Borel transform is then given by the series:

$$G(\xi) = \sum_{n \geq 1} \xi^{n-1} = \sum_{n \geq 0} \xi^n \,,$$

which converges for $|\xi| < 1$ its analytic continuation gives the meromorphic function:

$$G(\xi) = \frac{1}{1-\xi} \,.$$

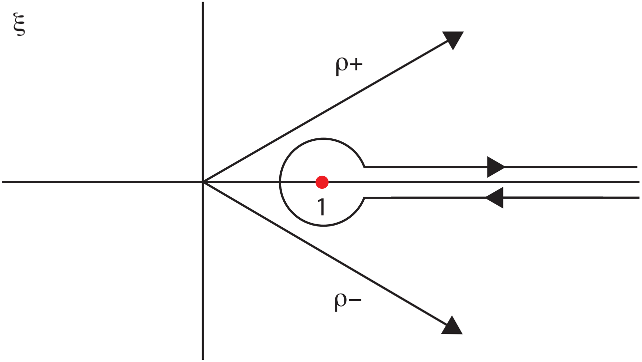
The Laplace transform gives the Borel sum:

$$\phi_\rho(z) = \int_\rho \frac{e^{-\xi/z}}{1-\xi} d\xi \,.$$

Stokes factors stem from residues

What is the difference of the analytic functions corresponding to different choices of ρ:

$$\phi_{\rho_+}(z) - \phi_{\rho_-}(z) = \int_{\mathcal{C}} \frac{e^{-\xi/z}}{1-\xi} d\xi = -2\pi i e^{-1/z} \,.$$



Resurgence builds an exciting new bridge between mathematics and physics

Content of the course

- Asymptotic expansions for ODEs and the Gamma function
- The complex geometry of Borel-Laplace summation
- Resurgence in exact WKB, triangulations of Riemann Surfaces
- Resurgence in Chern-Simons theory and topological string theory

Prerequisites

- Complex Analysis!!
- Differential Topology course recommended
- Some exposure to quantum mechanics is helpful but not necessary