Applications of Mathematics

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Overview of the theme

The theme can be divided into two broad categories:

- the formulation and analysis of *mathematical models*
- *methods or tools* needed in the process

Two modules per category; self-contained and independent.

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- **continuum mechanics**
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Two complementory sets of methods:

- **asymptotic and analytic**
- **numerical**

Again not comprehensive, but illustrative. In both cases the aim is to develop *approximate* methods in a systematic, quantitative way.

Continuum Mechanics (semester 1)

Continuum mechanics describes how deformable media behave.

Fluids (liquid/gas)
Solids (elastic/plastic)

Continuum hypothesis + Newton II

$O(10^{24})$ molecules: can't solve $F = ma$ for each

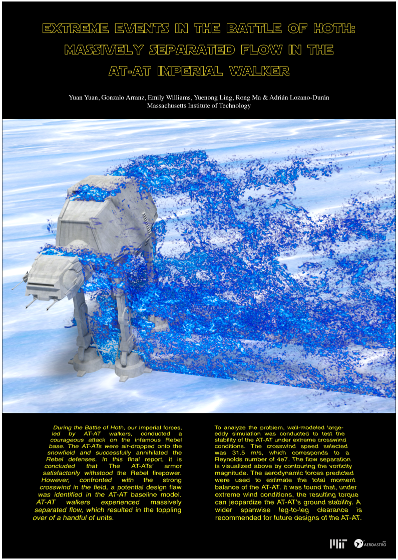
Treat medium as a continuum of parcels, each small on scale of motion, each containing large number of molecules

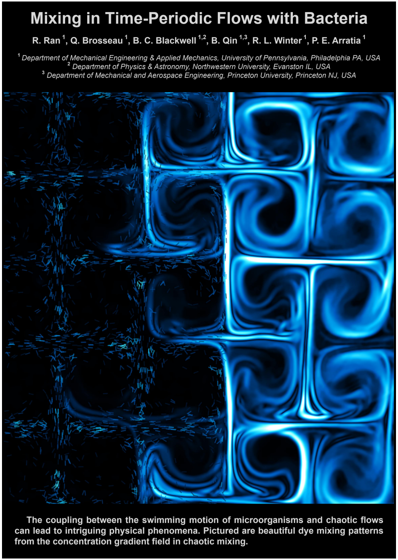
Then $\mathbf{u}$, p, ρ, etc, considered as functions of $(\mathbf{x}, t)$

Huge range of applications (nano-technology to astrophysics).

Diverse examples at: `https://gfm.aps.org/`

Continuum Mechanics (semester 1)





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 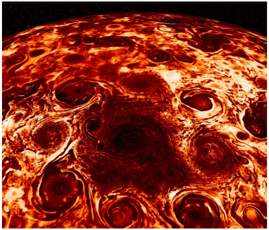 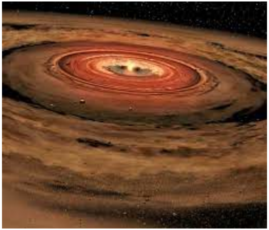

hurricane | jupiter pole | protoplanetary disc

all modelled with

$$q_t + J(\psi, q) = F \qquad q = \nabla^2 \psi$$

Continuum Mechanics (semester 1)

Topics:

- **continuum mechanics:** construction of dynamical models of deformable media
- **fluid dynamics:** lubrication theory, aerofoils, hydrodynamic stability
- **non-Newtonian fluids:** fluid viscosity depends on internal stresses

Mathematical Biology and Physiology (semester 2)

Mathematical modelling in the life sciences: exciting and rapidly evolving!

Modelling of just about any aspect of biological systems:

circulation
patterns
populations
cell dynamics
diseases/treatments

Mathematical Biology and Physiology (semester 2)

Topics:

- **bacterial resistance:** antibiotics
- **mathematical physiology:** microscale/macroscale & homogenisation
- **population modelling:** epidemiology, evolution, pathogen-host interactions, age-structured models
- **mathematical oncology:** cancer modelling

Methods for problems typically involving a *small parameter*, $\varepsilon \ll 1$

Difference between $\varepsilon = 0$ and $\varepsilon \ll 1$ is often profound.

Often remarkably successful, even when they shouldn't be!

Example: asymptotic expansion of the error function:

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$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{1}{5}\frac{x^5}{2!} - \ldots + \frac{(-1)^n}{2n+1}\frac{x^{2n+1}}{n!} + \ldots \right]$$

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- $x = 4$: needs > 40 terms for a reasonable approximation
- $x = 4$: first 16 terms sum to 10^5

Alternative approach: write

$$\mathrm{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

and integrate by parts n times:

$$\int_x^\infty e^{-t^2} dt = \int_x^\infty \frac{te^{-t^2}}{t} dt$$

$$= e^{-x^2} \left[\frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5} + \ldots + \frac{(-1)^{n-1}}{2^n} \frac{1.3.5...(2n-3)}{x^{2n-1}} \right] +$$

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- the radius of convergence is zero: it diverges for $x \neq 0$
- $x = 4$: first 2 terms give erf(4) to within 10^{-7}

Asymptotic and Analytical Methods (semester 1)

Topics:

- **multiple scales:** modulation and resonance
- **matched asymptotics:** boundary layers
- **WKB theory:** rapidly oscillating solutions, ray theory
- **approximation of integrals:** Watson's lemma, steepest descents
- **intermediate asymptotics:** self-similarity of solutions
- **resummation:** techniques for series

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Ways to solve mathematical models numerically.

Often there is a compromise between ease of implementation and efficiency.

Or between speed and accuracy, etc...

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But might find the method fails for $\alpha > \alpha_c$.
Try a relaxation:

$$\phi_{n+1} = (1-w)\phi_n + wL_\alpha^{-1}\left[q\mathcal{N}[\phi_n]\right]$$

Numerical Methods (semester 2)

Topics:

- **Ordinary DEs:** explicit and implicit methods (stability)
- **Stochastic DEs:** an introduction
- **Partial DEs:** finite-difference and finite-element methods
- **Linear Algebra:** linear systems, eigenvalues etc.

Prior knowledge, delivery and assessment

Assumed knowledge of standard, core undergraduate material:

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- complex variables

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Assessment: two assignments per module

- mix of analytic and computational work
- normally at least two weeks to complete each