

Applications of Mathematics

Theme Head: Richard Scott

School of Mathematics and Statistics
University of St Andrews

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Overview of the theme

The theme can be divided into two broad categories:

- the formulation and analysis of *mathematical models*
- *methods or tools* needed in the process

Two modules per category; self-contained and independent.

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- **continuum mechanics**
- **mathematical biology**

Not comprehensive, but illustrative of a range of approaches.

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Two complementary sets of methods:

- **asymptotic and analytic**
- **numerical**

Again not comprehensive, but illustrative. In both cases the aim is to develop *approximate* methods in a systematic, quantitative way.

Continuum Mechanics (semester 1)

Continuum mechanics describes how deformable media behave.

Fluids (liquid/gas)

Solids (elastic/plastic)

Continuum hypothesis + Newton II

$O(10^{24})$ molecules: can't solve $F = ma$ for each

Treat medium as a continuum of parcels, each small on scale of motion, each containing large number of molecules

Then \mathbf{u} , p , ρ , etc, considered as functions of (\mathbf{x}, t)

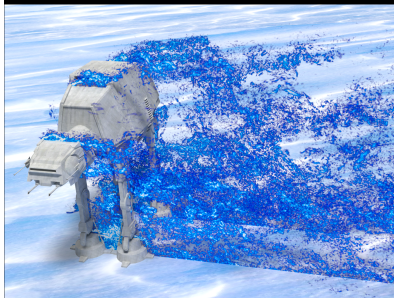
Huge range of applications (nano-technology to astrophysics).

Diverse examples at: <https://gfm.aps.org/>

Continuum Mechanics (semester 1)

EXTREME EVENTS IN THE BATTLE OF HOTH: MAXIMALLY SEPARATED FLOW IN THE AT-AT IMPERIAL WALKER

Yuan Yuan, Gonzalo Arraraz, Emily Williams, Yuanrong Ling, Boqiang Ma & Adrian Lucazo-Duran
Massachusetts Institute of Technology



During the Battle of Hoth, our Imperial forces, led by AT-AT walkers, conducted a successful attack on the Rebel base. The AT-ATs were air-dropped onto the planet and successfully annihilated the Rebel defenses. In this final report, it is concluded that the AT-ATs' armor solidly withstood the Rebel firepower. However, confronted with the strong crosswind in the field, a potential design flaw was identified in the AT-AT baseline model. AT-AT walkers experienced massively separated flow, which resulted in the toppling over of a handful of units.

To analyze the problem, well-motivated large-scale simulation was conducted to test the stability of the AT-AT under extreme crosswind conditions. The crosswind speed selected was 21.2 m/s, which corresponds to a Reynolds number of 467. The flow separation is visualized above by contouring the vorticity magnitude. The aerodynamic forces predicted were used to estimate the total moment balance of the AT-AT. It was found that, under extreme wind conditions, the resulting torque can jeopardize the AT-AT's ground stability. A wider spanwise leg-to-leg clearance is recommended for future designs of the AT-AT.



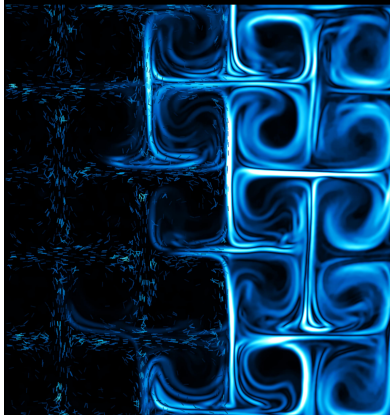
Mixing in Time-Periodic Flows with Bacteria

R. Ran¹, Q. Brosseau¹, B. C. Blackwell^{1,2}, B. Qin^{1,2}, R. L. Winter¹, P. E. Arratia¹

¹ Department of Mechanical Engineering & Applied Mechanics, University of Pennsylvania, Philadelphia PA, USA

² Department of Physics & Astronomy, Northwestern University, Evanston IL, USA

³ Department of Mechanical and Aerospace Engineering, Princeton University, Princeton NJ, USA



The coupling between the swimming motion of microorganisms and chaotic flows can lead to intriguing physical phenomena. Pictured are beautiful dye mixing patterns from the concentration gradient field in chaotic mixing.

<https://gfm.aps.org/>

Richard Scott, St Andrews

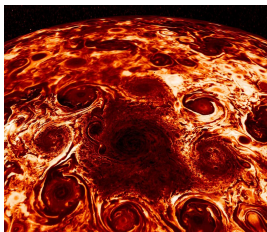
Applications of Mathematics



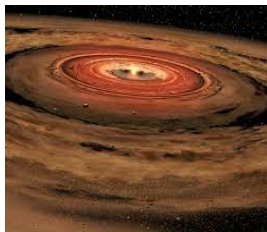
Continuum Mechanics (semester 1)



hurricane



jupiter pole



protoplanetary disc

all modelled with

$$q_t + J(\psi, q) = F \quad q = \nabla^2 \psi$$

Continuum Mechanics (semester 1)

Topics:

- **continuum mechanics:** construction of dynamical models of deformable media
- **fluid dynamics:** lubrication theory, aerofoils, hydrodynamic stability
- **non-Newtonian fluids:** fluid viscosity depends on internal stresses

Mathematical Biology and Physiology (semester 2)

Mathematical modelling in the life sciences: exciting and rapidly evolving!

Modelling of just about any aspect of biological systems:

circulation

patterns

populations

cell dynamics

diseases/treatments

Topics:

- **bacterial resistance:** antibiotics
- **mathematical physiology:** microscale/macroscopic & homogenisation
- **population modelling:** epidemiology, evolution, pathogen-host interactions, age-structured models
- **mathematical oncology:** cancer modelling

Asymptotic and Analytical Methods (semester 1)

Methods for problems typically involving a *small parameter*, $\varepsilon \ll 1$

Difference between $\varepsilon = 0$ and $\varepsilon \ll 1$ is often profound.

Often remarkably successful, even when they shouldn't be!

Asymptotic and Analytical Methods (semester 1)

Example: asymptotic expansion of the error function:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (< 1).$$

Seek an expansion to evaluate $\operatorname{erf}(x)$ for large x .

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- $x = 4$: needs > 40 terms for a reasonable approximation
- $x = 4$: first 16 terms sum to 10^5

Asymptotic and Analytical Methods (semester 1)

Alternative approach: write

$$\operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

and integrate by parts n times:

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- the radius of convergence is zero: it diverges for $x \neq 0$
- $x = 4$: first 2 terms give $\operatorname{erf}(4)$ to within 10^{-7}

Asymptotic and Analytical Methods (semester 1)

Topics:

- **multiple scales:** modulation and resonance
- **matched asymptotics:** boundary layers
- **WKB theory:** rapidly oscillating solutions, ray theory
- **approximation of integrals:** Watson's lemma, steepest descents
- **intermediate asymptotics:** self-similarity of solutions
- **resummation:** techniques for series

Numerical Methods (semester 2)

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Often there is a compromise between ease of implementation and efficiency.

Or between speed and accuracy, etc...

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Try a relaxation:

$$\phi_{n+1} = (1 - w)\phi_n + wL_\alpha^{-1}[q\mathcal{N}[\phi_n]]$$

Topics:

- **Ordinary DEs:** explicit and implicit methods (stability)
- **Stochastic DEs:** an introduction
- **Partial DEs:** finite-difference and finite-element methods
- **Linear Algebra:** linear systems, eigenvalues etc.

Prior knowledge, delivery and assessment

Assumed knowledge of standard, core undergraduate material:

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- linear algebra
- complex variables

More details on the module pages.

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Assessment: two assignments per module

- mix of analytic and computational work
- normally at least two weeks to complete each