Introduction to new course: "Conformal Field Theory and Vertex Operator Algebras"

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October 2, 2024
SMSTC symposium, Perth

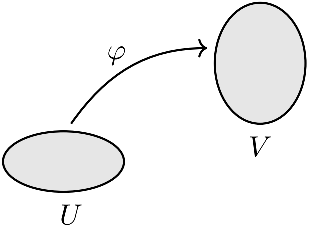
Conformal Field Theory = Quantum Field Theory invariant under conformal transformations

Conformal transformations

Consider a plane with a Riemannian metric

$$g = g_{\mu\nu} dx^\mu dx^\nu$$

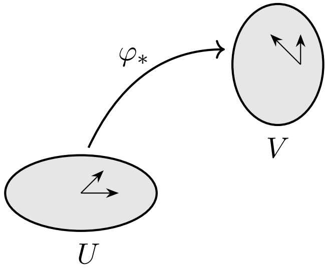
A conformal map of a domain $U \subset \mathbb{R}^2$ is a diffeomorphism $\varphi : U \to V$



for which

$$\varphi^* g = \Lambda g \qquad \Lambda = \Lambda(x) > 0$$

Conformal transformations in general do not preserve distances between points but preserve angles between vectors in tangent space



Rigid Motions

Rigid motions are conformal transformations. A classic theorem (Chasles ?) states that an orientation preserving rigid motion of the plain is a composition of a rotation and a translation

$$z \mapsto e^{i\alpha} z + B$$

and an orientation reversing rigid motion is a composition of a translation, rotation and a reflection

$$z \mapsto e^{i\alpha} \bar{z} + B$$

Affine Transformations

It is not hard to see that dilations

$$z \mapsto \rho z \,, \quad \rho > 0 \tag{1}$$

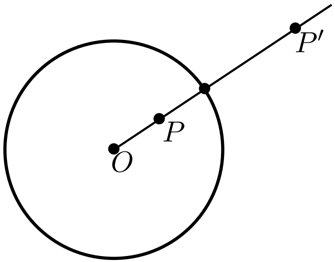
are conformal transformations. They can be combined with orientation preserving rigid motions to obtain complex Affine Transformations

$$z \mapsto Az + B \,, \quad A, B \in \mathbb{C} \,, \;\; A \neq 0$$

Affine transformations form a group.

Inversions

$$\mathcal{I} : z \mapsto \frac{R^2}{\bar{z}}$$



Inversions are conformal transformations defined on $\mathbb{C} \setminus \{\mathrm{pt}\}$. A holomorphic version

$$C \circ \mathcal{I} : z \to \frac{1}{z}\,, \quad R = 1\,.$$

Möbius transformations

Affine transformations plus the holomorphic inversion generate the group of Möbius transformations:

$$z \mapsto \frac{az+b}{cz+d}\,.$$

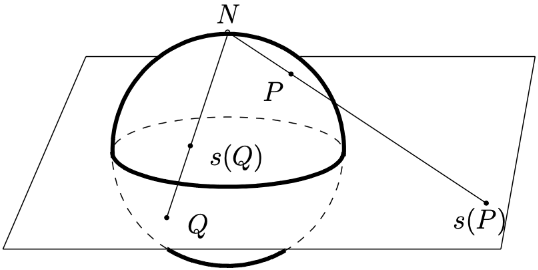
As a group it is isomorphic to a quotient group

$$\mathrm{PSL}(2,\mathbb{C}) := \mathrm{SL}(2,\mathbb{C})/\{\pm 1\} \tag{2}$$

where $\mathrm{SL}(2,\mathbb{C})$ is the group of complex 2x2 matrices with determinant 1.

Riemann sphere

While Möbius transformations are in general defined on $\mathbb{C} \setminus \{\mathrm{pt}\}$ they are extended to conformal bijections of the Riemann sphere that is a compactification of the complex plane.



Möbius transformations form the complete set of conformal orientation preserving transformations of the Riemann sphere=extended/compactified complex plane. If we drop the condition of the transformations being orientation-preserving we obtain a larger group of all conformal transformations of the extended plane that is the semidirect product

$$\mathrm{PSL}(2,\mathbb{C}) \rtimes \mathbb{Z}_2 \,, \qquad \mathbb{Z}_2 = \{1, C\}$$

Quantum Fields

A quantum field theory is a collection of functions

$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\ldots\mathcal{O}_n(z_n)\rangle : \mathbb{C}^n \to \mathbb{C}$$

that satisfy certain axioms. For example these functions are real-analytic and have singularities only when some of the arguments coincide. Key problems include finding sets of correlation functions satisfying the axioms and finding explicit expressions for these functions. This is a very difficult task in general with only asymptotic expansions available for the functions. Assuming these functions transform in a certain nice way under conformal transformations brings in many more relations.

Covariance under Möbius transformations

$$z \mapsto \varphi(z) = \frac{az+b}{cz+d}$$

$$\langle \mathcal{O}_1(\varphi(z_1)) \dots \mathcal{O}_n(\varphi(z_n)) \rangle$$
$$= |cz_1 + d|^{\Delta_1} \dots |cz_n + d|^{\Delta_n} \langle \mathcal{O}_1(z_1) \dots \mathcal{O}_n(z_n) \rangle$$

These equations have some resemblance to the transformation property of modular forms.

More structure comes from infinitesimal conformal transformations. Those are generated by vector fields

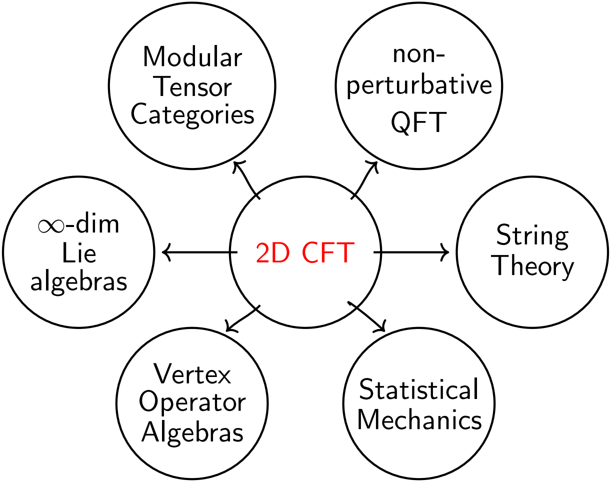
$$l_n = -z^n \partial_z - \bar{z}^n \partial_{\bar{z}} \quad n \in \mathbb{Z}\,, \ \ z \neq 0$$

They satisfy the commutation relations

$$[l_n, l_m] = (n-m) l_{n+m}\,, \quad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}$$

Each pair is called Witt algebra and is a complexification of the Lie algebra of the group of diffeomorphisms of a circle. In QFT one obtains a representation of its central extension called Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\delta_{n+m,0}}{12}(n^3 - n)\mathcal{C}\,, \quad [\mathcal{C}, L_n] = 0$$


Modular Tensor Categories
non-perturbative QFT
∞-dim Lie algebras
2D CFT
String Theory
Vertex Operator Algebras
Statistical Mechanics

Mathematics prerequisites for the course:

- Basic group theory
- Basic complex analysis (calculating contour integrals via residues)
- Basic functional analysis (Hilbert space, inner product, Riesz representation theorem, bounded, Hermitian linear operators)
- Basic differential geometry (tangent space, forms and vector fields, metric, push-forward and pull-back maps)
- Analysis (a notion of distribution, distributional derivative, Dirac's delta function)

Physics prerequisites

Physics prerequisites for the course:

- Some acquaintance with Quantum Mechanics is expected (Hilbert space, bra- and ket-vector notation, Hamiltonian, Schroedinger equation)
- No prior knowledge of quantum field theory is assumed. The course is intended to serve as an introduction to the subject taylored to mathematics students. The exposition is axiomatic in spirit but not overly formalised. Several general concepts and constructions will be explained: correlation functions, Ward identities, canonical quantisation, basic idea of functional integration, GNS construction of Hilbert space from correlation functions.

Course organisation

Course Notes:There are detailed course notes to be used together with the lectures. They are meant to complement the lectures with many additional details to be found in the notes. I suggest reading the material in the notes before and after the lectures. Also some additional reading is suggested in the notes.

Course organisation

Tutorials: There will be weakly tutorial sessions starting with week 2. Tutorial problems will be made available in advance and should be attempted **before** the tutorial session which will focus on the difficulties encountered by the students during the attempts. While the lectures will focus on conceptual issues and won't have many calculations done in detail the tutorial problems are meant to complement for that and provide practice needed to develop computational skills.

Course organisation

Assessment: Two take home assignments. One in the middle (worth 40%) and one at the end of the course (worth 60%).