SMSTC Opening Symposium
2-3 October 2024

# Gradient Flows

MAC-MIGS and SMSTC Advanced Course
Tuesdays 11.00-13.00, starting 22 October 2024
Bayes 5.46 and online

John Ball

# What is a gradient flow?

A gradient flow in $X = \mathbb{R}^n$ is an ODE of the form

$$\dot{x}(t) = -\nabla V(x(t)),$$

where $V \in C^1(\mathbb{R}^n)$.

Notice that for any solution

$$\frac{d}{dt}V(x(t)) = -|\nabla V(x(t))|^2,$$

so that $V(x(t))$ is nonincreasing along solutions, and constant for a solution if and only if $\dot{x}(t) = -\nabla V(x(t)) = 0$, that is $x(t) = p$ for some critical point $p$ of $V$.

Thus a gradient flow in $X = \mathbb{R}^n$ is a special case of a semiflow (dynamical system) endowed with a *Lyapunov function*, for which there is a general theory (which will be reviewed in the course) addressing the question of whether solutions converge to stationary points as $t \to \infty$.

However gradient flows have additional structure, which enables them to be defined and analyzed in Hilbert and Banach spaces, and surprisingly even in general metric spaces (a theory described in the 2008 book of Ambrosio, Gigli and Savare).

# Outline of course

- Review of semiflows with a Lyapunov function and convergence to rest points.
- Gradient flows in finite dimensions, elements of Morse theory, mountain pass theorem, Lojasiewicz inequality.
- Gradient flows in Hilbert Space. $\lambda-$convexity and the Brezis-Komura theorem.
- Gradient flows in metric spaces. Minimizing movement schemes.
- Wasserstein gradient flows and other applications to PDE.