#### SMSTC Supplementary module

# An introduction to Hopf algebras over fields

Andrew Baker (University of Glasgow)

last updated 01/10/2024

### Outline

**Prerequisites:** As background, basic knowledge of rings, modules and representation theory of groups would be useful, as well some familiarity with category theory, commutative diagrams and homological algebra.

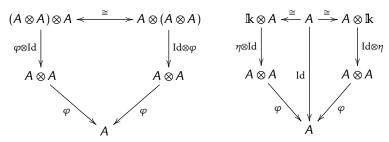
**Learning outcomes:** As well as exposure to important topics in modern abstract algebra, this is a good place to gain familiarity with the use of category theory in mathematics.

Who might be interested: Likely to be of interest to mathematicians working on topics such as representation theory of finite dimensional algebras and finite groups, knot theory, Lie theory, algebraic topology, algebraic geometry, non-commutative geometry.

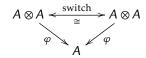
**Practicalities:** There will be opportunities for participants to give short talks on topics that particularly interest them, these could be used for gaining credit for the course.

# Quick introduction

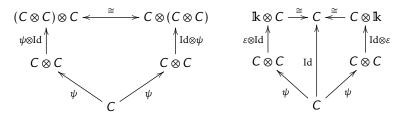
A &-algebra  $(A, \varphi, \eta)$  over a field & is a monoid in the monoidal category ( $\mathsf{Vect}_\&, \otimes$ ), i.e., a &-vector space A with a product  $\varphi \colon A \otimes A \to A$  and unit  $\eta \colon \& \to A$ , which make the following diagrams in  $\mathsf{Vect}_\&$  commute.



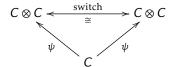
A is commutative if in addition the following diagram commutes.



The dual notion is that of a k-coalgebra, which is a triple  $(C, \psi, \varepsilon)$ , with C a k-vector space, a coproduct  $\psi \colon C \to C \otimes C$ , and a counit  $\varepsilon \colon C \to k$  fitting into the commutative diagrams shown.



This says that  $(C, \psi, \varepsilon)$  is a *comonoid* in **Vect**<sub> $\mathbb{R}$ </sub>. If the following diagram commutes then C is *cocommutative*.



A &-Hopf algebra H is a &-vector space which is both an algebra and a coalgebra together with an antipode  $\chi\colon H\to H$ , so that all of this structure interacts in a certain way. A Hopf algebra is a *group object* in the category of coalgebras or a *cogroup object* in the category of algebras, so Hopf algebras generalise groups!

#### Some examples:

- ▶ The group algebra  $\mathbb{k}[G]$  of a group G is a cocommutative Hopf algebra with the elements of G as a basis, product  $\varphi(g'\otimes g'')=g'g''$ , coproduct  $\psi(g)=g\otimes g$  and antipode  $\chi(g)=g^{-1}$ .
- ▶ If *G* is a compact Lie group or more generally an *H*-space,  $H_*(G; \mathbb{k})$  and  $H^*(G; \mathbb{k})$  are Hopf algebras.
- If  $\mathfrak g$  is a Lie algebra, its universal enveloping algebra  $U(\mathfrak g)$  is a cocommutative Hopf algebra.
- Affine group schemes have associated commutative Hopf algebras.
- Examples from combinatorics.
- Quantum groups are Hopf algebras which are neither commutative nor cocommutative.

## Outline of course content

Depending on time and the audience's interests, I expect to discuss most of the topics below.

- Some category theory: monoidal categories (vector spaces over a field as an important example), adjoint functors.
- Algebras and coalgebras over a field; bialgebras and Hopf algebras.
- Lots of examples.
- SubHopf algebras, adjoint actions and normal subHopf algebras.
- Modules and comodules, representation theory of a Hopf algebra.
- ► Hopf modules and finite dimensional Hopf algebras; every finite dimensional Hopf algebra is Frobenius.
- Quantum Groups.
- If time permits: Homological algebra for modules over Hopf algebras.