SMSTC Supplementary module

An introduction to Hopf algebras over fields

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Outline

Prerequisites: As background, basic knowledge of rings, modules and representation theory of groups would be useful, as well some familiarity with category theory, commutative diagrams and homological algebra.

Learning outcomes: As well as exposure to important topics in modern abstract algebra, this is a good place to gain familiarity with the use of category theory in mathematics.

Who might be interested: Likely to be of interest to mathematicians working on topics such as representation theory of finite dimensional algebras and finite groups, knot theory, Lie theory, algebraic topology, algebraic geometry, non-commutative geometry.

Practicalities: There will be opportunities for participants to give short talks on topics that particularly interest them, these could be used for gaining credit for the course.

Quick introduction

A $\Bbbk$-*algebra* (A, φ, η) over a field $\Bbbk$ is a *monoid* in the monoidal category $(\mathbf{Vect}_{\Bbbk}, \otimes)$, i.e., a $\Bbbk$-vector space A with a *product* $\varphi \colon A \otimes A \to A$ and *unit* $\eta \colon \Bbbk \longrightarrow A$, which make the following diagrams in $\mathbf{Vect}_{\Bbbk}$ commute.

$$
\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, leftrightarrow, "\cong"] \arrow[d, "\varphi\otimes\mathrm{Id}"'] & & A \otimes (A \otimes A) \arrow[d, "\mathrm{Id}\otimes\varphi"] \\
A \otimes A \arrow[dr, "\varphi"'] & & A \otimes A \arrow[dl, "\varphi"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes A \arrow[d, "\eta\otimes\mathrm{Id}"'] & A \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[dd, "\mathrm{Id}"'] & A \otimes \Bbbk \arrow[d, "\mathrm{Id}\otimes\eta"] \\
A \otimes A \arrow[dr, "\varphi"'] & & A \otimes A \arrow[dl, "\varphi"] \\
& A &
\end{tikzcd}
$$

A is *commutative* if in addition the following diagram commutes.

$$
\begin{tikzcd}
A \otimes A \arrow[rr, leftrightarrow, "\mathrm{switch}", "\cong"'] \arrow[dr, "\varphi"'] & & A \otimes A \arrow[dl, "\varphi"] \\
& A &
\end{tikzcd}
$$

The dual notion is that of a $\Bbbk$-*coalgebra*, which is a triple (C, ψ, ε), with C a $\Bbbk$-vector space, a *coproduct* $\psi \colon C \to C \otimes C$, and a *counit* $\varepsilon \colon C \to \Bbbk$ fitting into the commutative diagrams shown.

$$
\begin{tikzcd}
(C \otimes C) \otimes C \arrow[rr, leftrightarrow, "\cong"] & & C \otimes (C \otimes C) \\
C \otimes C \arrow[u, "\psi\otimes\mathrm{Id}"] & & C \otimes C \arrow[u, "\mathrm{Id}\otimes\psi"'] \\
& C \arrow[ul, "\psi"] \arrow[ur, "\psi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes C \arrow[r, "\cong"] & C & C \otimes \Bbbk \arrow[l, "\cong"'] \\
C \otimes C \arrow[u, "\varepsilon\otimes\mathrm{Id}"] & & C \otimes C \arrow[u, "\mathrm{Id}\otimes\varepsilon"'] \\
& C \arrow[ul, "\psi"] \arrow[ur, "\psi"'] \arrow[uu, "\mathrm{Id}"] &
\end{tikzcd}
$$

This says that (C, ψ, ε) is a *comonoid* in $\mathbf{Vect}_{\Bbbk}$.
If the following diagram commutes then C is *cocommutative*.

$$
\begin{tikzcd}
C \otimes C \arrow[rr, leftrightarrow, "\mathrm{switch}", "\cong"'] & & C \otimes C \\
& C \arrow[ul, "\psi"] \arrow[ur, "\psi"'] &
\end{tikzcd}
$$

A $\Bbbk$-*Hopf algebra* H is a $\Bbbk$-vector space which is both an algebra and a coalgebra together with an antipode $\chi \colon H \to H$, so that all of this structure interacts in a certain way. A Hopf algebra is a *group object* in the category of coalgebras or a *cogroup object* in the category of algebras, so Hopf algebras generalise groups!

Some examples:

- The *group algebra* $\Bbbk[G]$ of a group G is a cocommutative Hopf algebra with the elements of G as a basis, product $\varphi(g' \otimes g'') = g'g''$, coproduct $\psi(g) = g \otimes g$ and antipode $\chi(g) = g^{-1}$.
- If G is a compact Lie group or more generally an H-space, $H_*(G; \Bbbk)$ and $H^*(G; \Bbbk)$ are Hopf algebras.
- If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.
- Affine group schemes have associated commutative Hopf algebras.
- Examples from combinatorics.
- Quantum groups are Hopf algebras which are neither commutative nor cocommutative.

Outline of course content

Depending on time and the audience's interests, I expect to discuss most of the topics below.

- Some category theory: monoidal categories (vector spaces over a field as an important example), adjoint functors.
- Algebras and coalgebras over a field; bialgebras and Hopf algebras.
- Lots of examples.
- SubHopf algebras, adjoint actions and normal subHopf algebras.
- Modules and comodules, representation theory of a Hopf algebra.
- Hopf modules and finite dimensional Hopf algebras; every finite dimensional Hopf algebra is Frobenius.
- Quantum Groups.
- If time permits: Homological algebra for modules over Hopf algebras.