SMSTC Supplementary Modules:

Geometry of Gauge Fields
and
Classical and Quantum Integrable Systems

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Geometry of Gauge Fields

The inventors of gauge theory

Figure: James Clerk Maxwell and Hermann Weyl

A fruitful failure

1919. № 10.

ANNALEN DER PHYSIK.

VIERTE FOLGE. BAND 59.

1. *Eine neue Erweiterung der Relativitätstheorie; von H. Weyl.*

Kap. I. Geometrische Grundlage.

Einleitung. Um den physikalischen Zustand der Welt an einer Weltstelle durch Zahlen charakterisieren zu können, muß 1. die Umgebung dieser Stelle auf *Koordinaten* bezogen sein und müssen 2. gewisse *Maßeinheiten* festgelegt werden. Die bisherige Einsteinsche Relativitätstheorie bezieht sich nur auf den ersten Punkt, die Willkürlichkeit des Koordinatensystems; doch gilt es, eine ebenso prinzipielle Stellungnahme zu dem zweiten Punkt, der Willkürlichkeit der Maßeinheiten, zu gewinnen. Davon soll im folgenden die Rede sein.

Die Welt ist ein vierdimensionales Kontinuum und läßt sich deshalb auf vier Koordinaten $x_0\, x_1\, x_2\, x_3$ beziehen. Der Übergang zu einem anderen Koordinatensystem $\bar{x}_i$ wird durch stetige Transformationsformeln

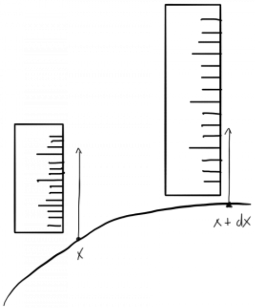
$$x_i = f_i(\bar{x}_0\, \bar{x}_1\, \bar{x}_2\, \bar{x}_3) \qquad (i = 0, 1, 2, 3) \tag{1}$$

vermittelt. An sich ist unter den verschiedenen möglichen Koordinatensystemen keines ausgezeichnet. Die Relativkoordinaten $d x_i$ eines zu dem Punkte $P = (x_i)$ unendlich benachbarten $P' = (x_i + d x_i)$ sind die Komponenten der infinitesimalen Verschiebung $\overrightarrow{P P'}$ (eines „Linienelementes" in P). Sie transformieren sich beim Übergang (1) zu einem anderen Koordinatensystem $\bar{x}_i$ linear:

$$d x_i = \sum_k \alpha_k^i\, d \bar{x}_k\,; \tag{2}$$

α_k^i sind die Werte der Ableitungen $\partial f_i / \partial \bar{x}_k$ im Punkte P. In der gleichen Weise transformieren sich die Komponenten ξ^i irgendeines *Vektors* in P. Mit einem die Umgebung von P bedeckenden Koordinatensystem ist ein „Achsenkreuz" in P

Length scale ('gauge') depends on position and time?



Assume the length scale is given by a positive, real-valued function $\ell : \mathbb{R}^4 \to \mathbb{R}^+$.

Length scale ('gauge') depends on position and time?

Parallel transport of a length scale in terms of 1-form
$A = A_t dt + A_1 dx_1 + A_2 dx_2 + A_3 dx_3$:

$$d\ell = -A\ell, \tag{1}$$

Change the gauge $\ell' = \lambda\ell$ with re-scaling function $\lambda : \mathbb{R}^4 \to \mathbb{R}^+$. In order to maintain the condition (1) in the new gauge we require

$$A' = A - d\ln\lambda.$$

$F = dA$ is unchanged!
Electromagnetic field? Einstein: ruled out by experiment

Introduce **gauge potential** a on $\mathbb{R}^4$ and **covariant derivatives**

$$D = d + A$$

Then the **gauged** Schrödinger equation

$$i\hbar D_t \psi = -\frac{\hbar^2}{2m} \sum_{j=1}^{3} D_j^2 \psi, \tag{2}$$

is **covariant** under **gauge transformation**

$$\psi \mapsto \psi' = e^{i\chi}\psi, \quad a \mapsto a - d\chi,$$

which leaves invariant the probability

$$p(t, R) = \int_R |\psi(t, \boldsymbol{x})|^2 d^3\boldsymbol{x}.$$

This is the gauge potential of **Maxwell's electrodynamics**!

What is gauge theory?

- All measurements depend on conventions and 'gauges' - but reality does not. Which mathematical quantities are gauge invariant?
- Gauge theories now used in physics, mathematics, economics and finance.
- The unreasonable effectiveness of gauge theories in modern physics and mathematics. Why?
- Here: gauge freedom captured by compact Lie groups $U(1), SU(2), SU(n)$....

Contents

1. Review of vector fields and differential forms on manifolds, introduction to Lie groups and Lie algebras.
2. Fibre bundles and associated vector bundles, connections, curvature, characteristic classes;
3. Maxwell theory as U(1) gauge theory, Dirac monopole as curvature, wave function of charged particle as section of associated line bundle;
4. Chern-Simons theory and the moduli space of flat connections on a Riemann surface, Atiyah-Bott symplectic structure;
5. Classical Yang-Mills theory, monopoles and instantons, self-duality equations, ADHMN construction of instantons and monopoles;
6. Outlook on moduli spaces of instantons and monopoles, S-duality and L^2-cohomology.

A giant of modern gauge theory

Figure: Michael Atiyah with statue of James Clerk Maxwell in Edinburgh

Prerequisites

- Some understanding of differentiable manifolds and differential forms, group theory
- Some familiarity with Lie algebras and Lie groups would be helpful but will **not** be assumed.

Take this course if you are interested in ...

- Connections between geometry, topology and physics,
- The mathematical theory of fibre bundles and connections,
- The language in which the Standard Model of Particle Physics is formulated,
- Beautiful applications of mathematics to physics: Yang-Mills theory, magnetic monopoles, instantons.
- Surprising applications of physics to mathematics: Donaldson theory, knot invariants from Chern-Simons theory, Seiberg-Witten theory (but we will not study them here in any detail!).

The team

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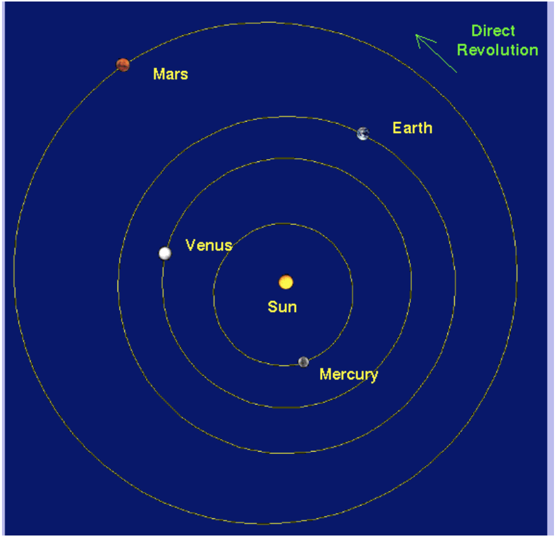
Classical and Quantum Integrability

The oldest classical integrable system: Kepler's problem



Classical integrability in a nutshell

We have

- A symplectic manifold M of dimension $2n$: physically the phase space or space of 'positions and momenta'
- Poisson brackets $\{f, g\}$ for $f, g \in C^\infty(M)$
- A Hamiltonian $H \in C^\infty(M)$ which generates time evolution according to
$$\dot{f} = \{H, f\}.$$

We want:

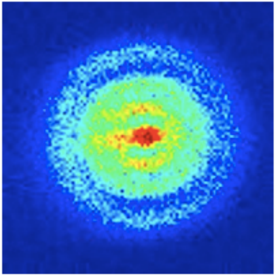
- $n-1$ further functions $f_1, \ldots, f_{n-1}$ so that
$$\{H, f_i\} = 0, \quad \text{and} \quad \{f_i, f_j\} = 0, \quad i, j = 1, \ldots, (n-1).$$
- An understanding of the geometrical flow generated by H.

Contents

1. **Foundations**: Review of manifolds, differential forms, vector fields and Lie derivatives; Lie groups and Lie algebras.
2. **Introduction to symplectic geometry and mechanics**: Hamiltonian mechanics; Poisson brackets; symmetry and Noether's theorem in Hamiltonian mechanics; moment(um) maps; Liouville theorem; examples
3. **Poisson-Lie structures I**: Poisson manifolds and symplectic leaves; co-adjoint orbits; Poisson-Lie algebras and Poisson-Lie groups; examples.
4. **Poisson-Lie structures II**: Co-boundary Poisson-Lie algebras and the classical Yang-Baxter equations; classical doubles; Sklyanin brackets; dressing action and symplectic leaves; examples.
5. **Classical integrable systems:** Lax pairs and classical r-matrices; construction of integrable systems out of co-boundary Lie bi-algebras; applications (e.g Toda chain)

Quantum integrability: the hydrogen atom

Orbital structure of H excited state



We have

- A Hilbert space $\mathcal{H}$: the space of quantum states.
- Self-adjoint operators $A : \mathcal{H} \to \mathcal{H}$: the observables of the theory.
- A particular self-adjoint operator, called the Hamiltonian H, which generates time evolution according to

$$\dot{A} = i\hbar[H, A].$$

We want:

- A (possibly infinite) family operators A_i, $i = 1, 2 \ldots$ which satisfy

$$[A_i, A_j] = [A_i, H] = 0, \quad i, j = 1, \ldots$$

- The ground state of H
- The discrete spectrum of H and the A_i, $i = 1, 2 \ldots$
- Other quantum properties of the system such as correlation functions.

Contents

1. **Statistical lattice models & combinatorics:** Motivating examples from enumerative combinatorics; partition functions; transfer matrices; quantum R-matrices; the quantum Yang-Baxter equation;
2. **The Yang-Baxter Algebra & the Bethe Ansatz:** Monodromy matrices and the Yang-Baxter algebra; diagonalising the transfer matrix; Bethe equations; guiding example: the 6-vertex model on the cylinder and the torus
3. **Quantum Groups I:** Axioms and examples; the Yang-Baxter algebra as a quantum group; the 6-vertex model and XXZ spin-chain as motivating example
4. **Quantum Groups II:** Representation theory; $U_q(\widehat{\mathfrak{sl}}_2)$; the R-matrix; short-exact-sequences and the XXZ fusion hierarchy

Prerequisites

- Some understanding of differentiable manifolds and differential forms, group theory
- Some familiarity with Lie algebras and Lie groups would be helpful but will **not** be assumed.

Take this course if you are interested in ...

- A rapidly evolving field which connects algebra, geometry, topology and physics
- A geometrical formulation of classical mechanics in the language of symplectic and Poisson structures
- The classical Yang-Baxter equation and how it gives rise to Poisson-Lie structures and integrable systems
- The role of quantum groups in quantum integrable systems
- A toolkit for solving problems in quantum field theory, string theory and condensed matter physics which are not (easily) amenable to other techniques

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