Fourier & harmonic analysis :-

Time : Mondays 11:15 - 12:45.

Assessment: Light assessment (2 x homework handins) for students taking course for credit.

Notes: Comprehensive set of course notes w/ exercises.

Overview:-

I Definition of Fourier series / Fourier transform + Convergence problems!

II Maximal functions + pointwise almost everywhere convergence

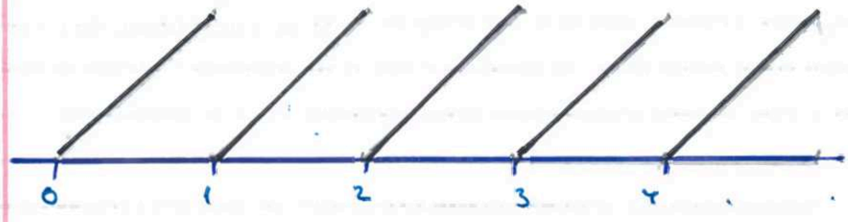
III Hilbert transform + Calderón - Zygmund theory

IV Littlewood - Paley theory.

I. Def^{n} of Fourier series / Fourier transform.

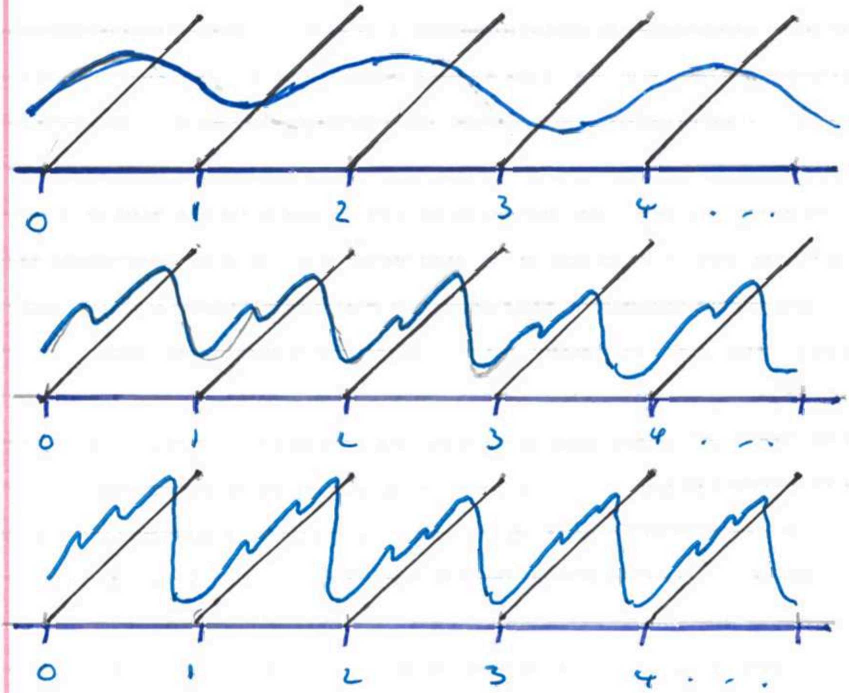
• Basic principle of Fourier series :-

$f : \mathbb{R}^n \to \mathbb{C}$ 1-periodic



can be written as a superposition of "harmonics"

$$f(x) = \sum_{k \in \mathbb{Z}^n} a_k \, e^{2\pi i k \cdot x}, \qquad a_k \in \mathbb{C}$$



$$a_k := \int_{[0,1)^n} e^{-2\pi i x \cdot k} f(x)\, dx \qquad \left(=: \hat{f}(k) \right)$$

"Fourier coefficients".

• Fourier transform :-

$f : \mathbb{R}^n \to \mathbb{C}$ integrable (say)

Fourier inversion formula :-

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \hat{f}(\xi)\, d\xi$$

where

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x)\, dx$$

is the Fourier transform.

• Convergence problems :-

Understand to what extent

$$f(x) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{2\pi i x \cdot k} \quad \text{(Fourier series)}$$

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \hat{f}(\xi)\, d\xi \quad \text{(Fourier inversion)}$$

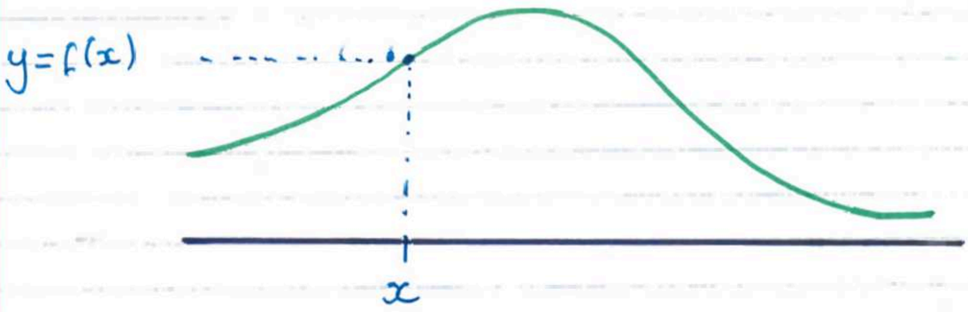
rigorously hold.

- Modes of convergence
  - Norm convergence (in L^p)
  - Pointwise almost everywhere convergence.
- Minimal hypotheses on f
  - Size conditions rather than regularity/smoothness conditions.

III Hilbert transform + Calderón–Zygmund theory.

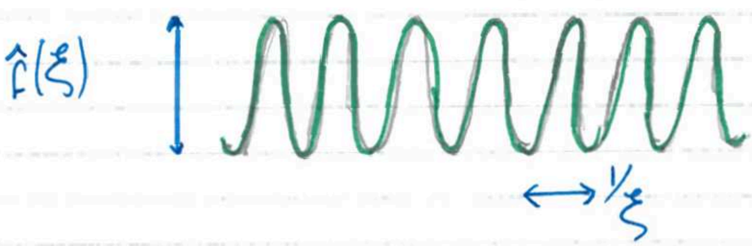
Fourier transform $\mathcal{F} : f \longmapsto \hat{f}$

- f is the physical / spatial rep. of the function.



"at point x the function has value $y=f(x)$".

- $\hat{f}$ is the frequency rep. of the function



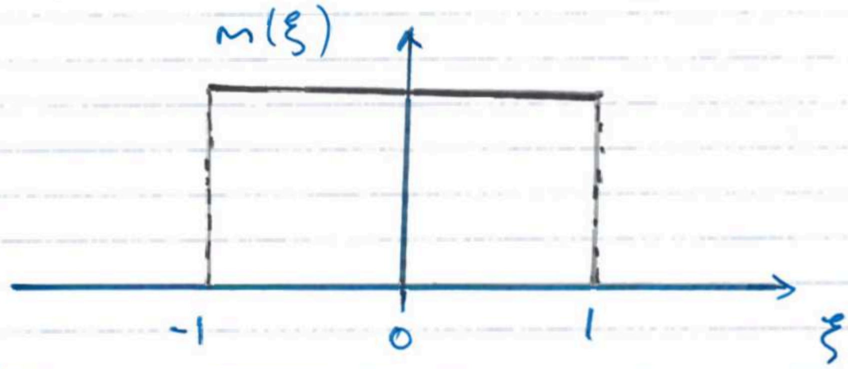
"the portion of f which oscillates at frequency ξ has amplitude $\hat{f}(\xi)$".

Fourier multipliers

$$f \xrightarrow{\mathcal{F}} \hat{f} \xrightarrow{m\cdot} m\cdot\hat{f} \xrightarrow{\mathcal{F}^{-1}} (m\cdot\hat{f})^{\vee}$$

pass to freq rep. — manipulate freq. — return to spatial rep.

Example :- $m(\xi) := \chi_{[-1,1]}(\xi) = \begin{cases} 1 & \text{if } -1 \leq \xi \leq 1 \\ 0 & \text{otherwise} \end{cases}$



multiplier operation

$$f \underset{\mathcal{F}}{\longmapsto} \hat{f} \underset{\chi_{[-1,1]}\cdot}{\longmapsto} \chi_{[-1,1]} \cdot \hat{f} \underset{\mathcal{F}^{-1}}{\longmapsto} (\chi_{[-1,1]} \hat{f})^{\vee}$$

corresponds to a low pass filter

On the physical side, can be represented in terms of the Hilbert transform :-

$$Hf(x) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x-y)}{y} \, dy$$

- convolution with kernel $\frac{1}{\pi y}$.
- Singular integral operator.