Analysis theme: pure and applied

Heiko Gimperlein h.gimperlein@hw.ac.uk

Heriot-Watt University

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2 pure modules:

- Measure and Integration (Semester I)
- Functional Analysis (Semester II)

2 applied modules:

- Dynamical Systems and Conservation Laws (Semester I)
- Elliptic and Parabolic PDEs (Semester II)

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- Integration as a tool Geometry, representation theory, probability, applied analysis etc.
- Geometry Gauss Bonnet Theorem

$$\int_M K dA = 2\pi \chi(M)$$

- Applied analysis solving differential equations
- Representation theory: average over compact group by integrating over the group.

Riemann integral:

need to know measure of intervalls / cubes [a,b]ⁿ



Lebesgue integral: need to know measure of sublevel sets $\{x : f(x) < a\}$

sublevel sets can be wild:



Sublevel sets can be wild Can we define their volume?





There exist subsets of Rⁿ for which volume can't be defined Banach-Tarski paradox decomposes ball into two balls, using translations and rotations (using bizarre pieces)



Analysis

Abstract Measure and Integration

- Abstract measure spaces and Lebesgue integral
- Setting for Probability theory
 - A measurable set E corresponds to an event
 - A measurable function corresponds to a random variable
- Two lectures (foundations and complex measures)

This abstract framework will unify:

- Riemann and Lebesgue integral on \mathbb{R} : $\int_{\mathbb{R}} f(x) dx$
- Infinite series: $\sum_{n=1}^{\infty} a_n$
- Elements of $C(X)^*$ (the continuous dual space of C(X)):

$$\Lambda \in \mathcal{C}(X)^* \implies \Lambda(f) = \int_X f(x) d\mu(x)$$

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Constructing measures

- Construction of Lebesgue measure on real line
- Carathéodory construction.
 - using outer measures
 - fractal sets and Hausdorff dimensions
- Product measures: $X = Y \times Z$ (e.g. \mathbb{R}^2
 - Fubini's theorem: when is it legitimate to change the order in a double integral.
- Radon measures dual space of C(X)
- End with discussion of Fourier series and overview of where research goes from here.

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Functional analysis:

Vector spaces and operators in infinite dimensions



Provides tools used right across mathematics: Language of analysis

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Provides tools used right across mathematics: Language of analysis

I'm not an analyst, why should I care?

- Banach algebras in algebraic geometry (perfectoid spaces, Fields medal Peter Scholze 2018)
- Hilbert and Banach spaces in numerical finite element methods
- quantum mechanics = Hilbert spaces, spectral theory, C* algebras

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Banach and Hilbert spaces

 $L^{p}(X)$ spaces, C(X), $C(X)^{*}$, etc ... Setting for Fourier series

- Linear operators and linear functionals
- Fundamental theorems: in the 'Scottish book' from cafe in Lwów. Baire Category, Open Mapping, Uniform Boundedness Principle, etc...
- Three lectures

Dual spaces and weak topologies

- Weak and weak* topologies
 - Banach-Alaoglu (weak* compactness of unit ball)
- Convexity in infinite dimensional setting.
 - Krein-Milman Theorem (very useful convexity result)
 - Use to produce Haar measure on compact groups.
- Two lectures

- Compact operators and their spectra
- General spectral theory

Banach algebras Spectral theorem for self-adjoint operators

- Commutative C*-algebras Gelfand's theory
- Four lectures; again aiming to finish with overview lecture pointing towards current research in this direction.

Formal prerequisites for both pure courses

- Undergraduate real analysis
 - Sequences, series, pointwise and uniform convergence.
 - Continuous and differentiable functions, basic properties.
- Metric space topology (at least in \mathbb{R}^d)
 - continuity of functions, open, closed and compact sets
- Countablilty
 - A set *S* is **countable** if $S = \{s_1, s_2, s_3, ...\}$.
 - The set of reals \mathbb{R} is uncountable!

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For: Measure and Integration

- Will start with full account of Riemann integral, so self contained.
- If you've not seen the Riemann integral (or a version of the Lebesgue integral on ℝ) before, you'll need to be motivated.

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For: Functional Analysis

- If seen measure theory and some Fourier analysis before ideal!
- Without measure theory, should be able to get a lot out of the course, but need to ignore some of the examples.

Assessment for both courses

Ongoing feedback

- Exercises associated to each lecture.
- Not part of formal assessment; for discussion in tutorials.
- Some lectures (certainly including me) will ask for groups to present some of these in lectures.

Formal assessment for each course

- Two assignments for each course; normally consisting of three questions per assessment.
- In particular, there'll be no measure theory assumed in the functional analysis assessment

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Partial differential equations (PDE)

Study equations $\mathcal{F}(x, y, \dots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \dots, \frac{\partial^2 u}{\partial x \partial y}, \dots) = 0$ for a function $u = u(x, y, \dots)$, where \mathcal{F} is known and u is not.

Classical descriptions of natural phenomena: • d'Alembert 1752, 1–dim. wave equation for vibrating string: $\partial_t^2 u(t, x) - \partial_x^2 u(t, x) = f(t, x)$

• Euler 1759, D. Bernoulli 1762, wave equation for acoustics: $\partial_t^2 u(t, \vec{x}) - \Delta u(t, \vec{x}) = f(t, \vec{x}), \qquad \Delta = \partial_{x_t}^2 + \dots + \partial_{x_t}^2$

• Laplace 1780, Laplace equation for gravitational potential field $-\Delta u(\vec{x}) = f(\vec{x})$

Fourier 1822, heat equation

 $\partial_t u(t, \vec{x}) - \Delta u(t, \vec{x}) = f(t, \vec{x})$

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Some modern applications



Heiko Gimperlein (Heriot-Watt University)

Analysis

Dynamical systems and conservation laws Lorentz system of ODEs: weather forecasts



ODE Example

Lorenz Equations

$$\frac{dx}{dt} = \sigma(y-x)
\frac{dy}{dt} = rx - y - xz
\frac{dz}{dt} = xy - bz$$
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where σ, r, b are positive parameters.

Simple looking system of ODEs derived by Lorenz to help analyse theoretical problems in meteorology and weather prediction.

Based on a simplified model of convection: when a fluid is heated from below.

Let's look at what happens to **same** initial data as *r* is increased.

Lorenz Equs: $r = 10, \sigma = 10, b = 8/3$



Solution converges to a 'fixed point'.

Lorenz Equs: r = 24.05, $\sigma = 10, b = 8/3$



Solution converges to a 'periodic orbit'.

Lorenz Equs: r = 28, $\sigma = 10, b = 8/3$



Solution converges to classic 'chaotic attractor'.

Summary: Dynamical systems and conservation laws

- Dynamical systems and bifurcation
- Scalar conservation laws
- Systems of hyperbolic PDEs and shock waves

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Some basic questions

• Modelling:

What are the equations and where do they come from? *choose other modules*

 ∃ solution? Uniqueness? Dependence on data? existence of classical solutions for Laplace and heat equations (applied treatment)
 Sobolev spaces and generalized weak solutions

Qualitative properties?

self-similar solutions and travelling waves, maximum principles, Gidas-Ni-Nirenberg theorem on symmetry of solutions

• numerical approximation mathematical background behind finite elements, for numerical methods choose other modules

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Formal prerequisites for both applied courses

- undergraduate ordinary differential equations
- single- and multivariable real analysis, linear algebra
- Semester II more or less self-contained, but Semester I useful for motivation and background.

Assessment for both applied courses

Ongoing feedback

- Exercises associated to each lecture.
- Not part of formal assessment; for discussion in tutorials.
- Some lecturers (including me) might ask for groups to present some of these in lectures.

Formal assessment for each course

around 8 questions each semester, divided over 1 or 2 assignment sheets.

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Conclusion

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Interactions:

Pure maths is being applied, applied math solves pure problems.