SMSTC: Probability and Statistics

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Outline

- Probability and Statistics
- Course outlines and teaching teams
- Prerequisites
- Assessment
- Feedback

Probability and Statistics

"Chance, too, which seems to rush along with slack reins,
is bridled and governed by law."

– Boethius (ca. 480–505),
The Consolation of Philosophy

Probability and Statistics

- mathematical modelling of uncertainty: random events and random processes evolving in time.
- strongly driven by experimental observation, physical intuition, and ideas of information evolving in time.
- crucial to understand dependence between different elements of our model.

Probability and Statistics

Probability

- Building and analysing mathematical models of randomness, using elements of measure theory, functional analysis, combinatorics.
- Models include parameters, which can be specified in particular applications.

Statistics

- Model fitting from experimental data: How do we choose select the correct model? How do we fit parameters to a given data set? How do we handle imperfect (missing/contaminated/...) data? How do we quantify uncertainty in our estimates?
- Testing plausibility of given conjectures.
- Simulation of intractable probability distributions.

Foundations of Probability (Semester 1)

A gambler starts with £X_0. At turn $n = 1, 2, \ldots$, he stakes £S_n, and

- gains £S_n with probability $p > 1/2$, or
- loses £S_n with probability $1 - p$.

We let £X_n be his total wealth after turn n, and assume (reasonably!) that $0 \leq S_n \leq X_{n-1}$.

How can the gambler maximize his long-term gain?

Calculations using *conditional expectation* show that $E(X_n)$, the gambler's average wealth after turn n, is maximised by choosing $S_n = X_{n-1}$. But, this is not a viable long-term strategy (what happens the first time you lose?)...

If we instead try to maximise $E\log(X_n)$, we can show that this is achieved using the strategy $S_n = (2p-1)X_{n-1}$.

One way to do this is to show that a certain linear shift of $\log(X_n)$ is a *martingale* in this case, and a *supermartingale* in all others.

We can also check, using the *law of large numbers*, that if

- our gambler uses this strategy, and has £X_n after tun n, and
- another gambler uses the strategy $\widetilde{S}_n = \lambda\widetilde{X}_{n-1}$ (where $\lambda < 1$ and $\lambda \neq 2p-1$), and has £$\widetilde{X}_n$ after turn n

then $X_n/\widetilde{X}_n$ grown exponentially for large n, with probability 1. Hence, the choice $\lambda = 2p-1$ is a better choice than any other.

Foundations of Probability (Semester 1)

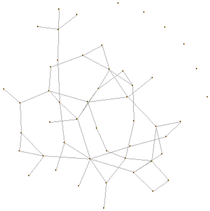
- **Fundamentals**: probability spaces, σ-algebras, probability measures, conditioning and independence
- **Random variables** and their distributions, important special distributions (binomial, Poisson, geometric, normal, exponential etc.)
- **Convergence** and **limit theorems**
- **Conditional expectation** and **martingales**
- **Renewal theory**

Stochastic Processes (Semester 2)

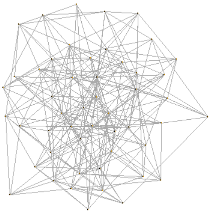
Suppose we have n vertices/nodes.

Each pair of vertices is joined by an edge/link with probability p, independently of all other pairs of vertices.

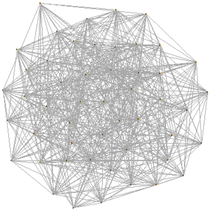
This is the Erdős–Rényi random graph $G(n, p)$. It can be used to model a 'typical' (or 'unstructured' or 'random') communication (or power, or distribution, or ...) network, for example.



$p = 0.05$



$p = 0.2$



$p = 0.5$

Let $p = c/n$. Then (under some mild conditions on c) $G(n, p)$ contains a path of length at least constant $\times$ n with probability 1, for large enough n.

This is is proved by analysing an algorithm which explicitly constructs such a path, and exploiting the *Markovian* structure present in the algorithm.

Let K_n be the complete graph, with n vertices and an edge between each pair of vertices. Suppose we colour each edge of K_n either red or blue.

There is a colouring of K_n which contains at most $\binom{n}{a}2^{1-\binom{a}{2}}$ monochromatic copies of the complete graph K_a.

We can prove this by

- Randomly colouring K_n (each edge is red with probability $1/2$, or blue otherwise, independently of the other edges);
- Calculating that the average number of monochromatic copies of K_a is $\binom{n}{a}2^{1-\binom{a}{2}}$; and
- Concluding that there must exist a colouring with at most this many monochromatic copies of K_a.

Stochastic Processes (Semester 2)

- **Markov chains** and **processes**, **Poisson processes**
- **Applications**, including connections to statistics and graph theory
- **Brownian motion** and **stochastic calculus**

Probability: Teaching team

- Burak Buke (Edinburgh)
- Damian Clancy (Heriot-Watt)
- James Cruise (Heriot-Watt)
- Fraser Daly (Heriot-Watt)
- Sergey Foss (Heriot-Watt)
- István Gyöngy (Edinburgh)
- Michela Ottobre (Heriot-Watt)
- David Siska (Edinburgh)

Probability: Prerequisites

- Elements of mathematical analysis, linear algebra and combinatorics at undergraduate level.
- For Stochastic Processes, in addition: Probability theory, either at undergraduate level or from Foundations of Probability.
- The ability to think both rigorously and intuitively!

Probability: Assessment

Each module is assessed by two written assignments.

Provisional deadlines on:

- Foundations of Probability: 20 November 2018 and 8 January 2019.
- Stochastic Processes: 19 February 2019 and 2 April 2019.

Assignments will be available at least two weeks before the deadline.

Solutions for (at least) one assignment from each module should be prepared using LaTeX.

Linear model:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \epsilon_i \,,$$

for $i = 1, \ldots, n$ (where n is the sample size), and where $\epsilon_1, \ldots, \epsilon_n$ are independent and identically distributed with $\epsilon_1 \sim \mathrm{N}(0, \sigma^2)$.

More succinctly

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \,, \qquad \boldsymbol{\epsilon} \sim \mathrm{N}(\mathbf{0}, \sigma^2 \mathbf{I}) \,.$$

Residual Sum of Squares:

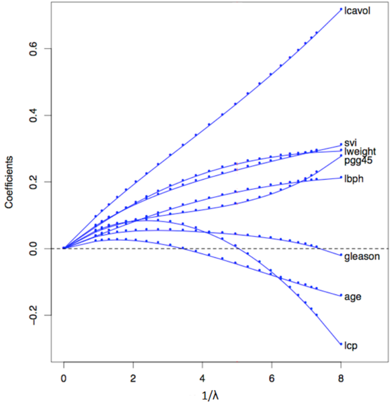
$$\mathrm{RSS} = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \,,$$

minimized by choosing

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} \,.$$

What happens when $\mathbf{X}^T\mathbf{X}$ is singular?
One possible solution: Ridge regression

$$\widehat{\beta}^{\text{ridge}} = (\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathbf{y}\,.$$



Regression and Simulation Methods (Semester 1)

- **Introduction to R**
- **Linear models**: Estimation, testing, model checking, factors, model fitting in R. Analysis of simple designed experiments. Case studies.
- **Likelihood and optimisation**: Likelihood principles and key distributional results. Examples. Newton's method for optimisation. Two-parameter likelihoods. general optimisation methods. Implementation in R.
- **Generalised linear models**: Exponential family. Link functions. Examples. Iteratively weighted least squares. Model fitting in R. Case studies.
- **Simulation and bootstrapping**: Non-parametric bootstrap; confidence intervals; implementation in R. Parametric bootstrap. Simulation methods and implementation in R.
- **Case study**

Regression and Simulation Methods (Semester 1)

The first half of Regression and Simulation Methods will be run as an online audio/video course. It cover what for many will be revision, and this flexible form of delivery allows participants to study different parts of the material at a speed and depth appropriate for them.

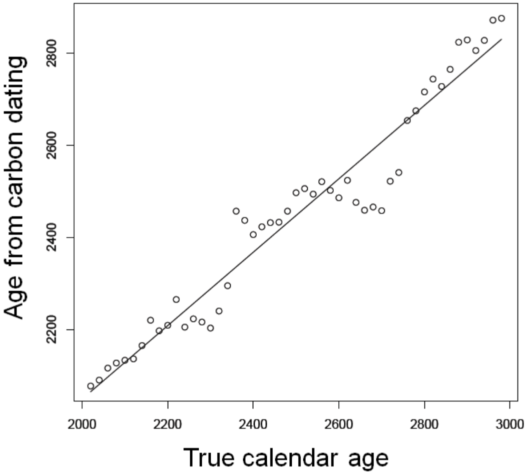
We ask you to check the course materials on the SMSTC website. If any of it is unfamiliar, you can view the relevant lectures, and attempt the related tutorial questions.

Tutorial support will be arranged locally.

Regular videoconferencing sessions will begin in the sixth session (13 November).

Modern Regression and Bayesian Methods (Semester 2)

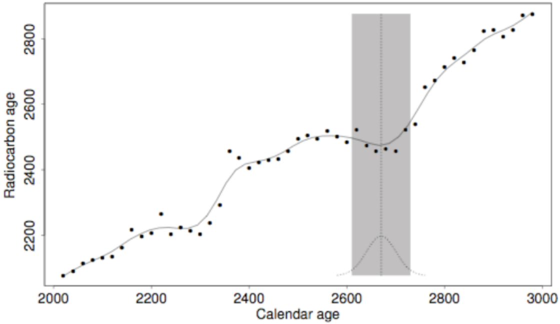
Radiocarbon data: high precision measurements of Carbon-14 in Irish oak, used to construct a calibration curve (here with line of best fit)



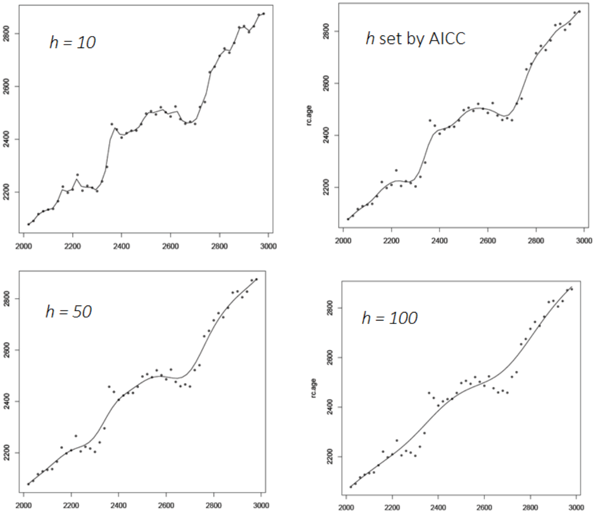
One solution to non-linearity: *local* linear regression. Solve

$$\min_{\alpha,\beta} \sum_{i=1}^{n} \{y_i - \alpha - \beta(x_i - x)\}^2 \, w(x_i - x; h)\,,$$

for a weight function w, and take $\widehat{\alpha}$ as the estimate at x.



We have a choice of the parameter h:



Modern Regression and Bayesian Methods (Semester 2)

- **Random effects models**: Methods for linear and non-linear mixed effects models. Case studies.
- **Modern regression**: Density estimation. Non-parametric regression. Bandwidth selection. Examples. Additive models. The backfitting algorithm. Examples.
- **Bayesian methods**: Priors and posteriors. Prior sensitivity. Marginal distributions.
- **Markov chain Monte Carlo**: Metropolis-Hastings algorithm. Gibbs sampler. Convergence, burn-in, mixing properties, tuning parameters. WinBUGS. MCMC simulations in R. Examples. Advanced topics: eg, random effects, missing data, model selection.
- **Case study**

Statistics: Teaching team

- Adrian Bowman (Glasgow)
- Valentin Popov (St Andrews)
- Serveh Sharifi Far (Edinburgh)
- TBC

Statistics: Prerequisites

- Basic concepts in probability (elementary probability distributions), statistics (idea of estimation, confidence intervals, hypothesis tests), calculus, and linear algebra. These would usually be provided in first undergraduate courses.
- For Modern Regression and Bayesian Methods: the semester 1 course (Regression and Simulation Methods), or equivalent.

Statistics: Assessment

Regression and Simulation Methods:

- One written assignment (based on the final five lectures), deadline in early January. The assignment will be available by mid-December.

Modern Regression and Bayesian Methods:

- Two written assignments, one after each block of five lectures. Assignments will be available at least two weeks before the deadline.

Feedback

- is a two-way process.
- if you have any questions/concerns, get in touch with me (`f.daly@hw.ac.uk`, 0131 451 3212) or another member of the teaching team.
- feedback and questions are encouraged during lectures.
- please don't wait for the end of the course!