SMSTC Supplementary module

An introduction to Hopf algebras over fields

Andrew Baker (University of Glasgow)

last updated 01/10/2023

Outline

Prerequisites: As background, basic knowledge of rings, modules and representation theory of groups would be useful, as well some familiarity with category theory, commutative diagrams and homological algebra.

Learning outcomes: As well as exposure to important topics in modern abstract algebra, this is a good place to gain familiarity with the use of category theory in mathematics.

Who might be interested: Likely to be of interest to mathematicians working on topics such as representation theory of finite dimensional algebras and finite groups, knot theory, Lie theory, algebraic topology, algebraic geometry, non-commutative geometry.

Practicalities: There may be an opportunity for participants to give short talks on topics that particularly interest them; this can be discussed during the course.

Outline

Prerequisites: As background, basic knowledge of rings, modules and representation theory of groups would be useful, as well some familiarity with category theory, commutative diagrams and homological algebra.

Learning outcomes: As well as exposure to important topics in modern abstract algebra, this is a good place to gain familiarity with the use of category theory in mathematics.

Who might be interested: Likely to be of interest to mathematicians working on topics such as representation theory of finite dimensional algebras and finite groups, knot theory, Lie theory, algebraic topology, algebraic geometry, non-commutative geometry.

Practicalities: There may be an opportunity for participants to give short talks on topics that particularly interest them; this can be discussed during the course.

Outline

Prerequisites: As background, basic knowledge of rings, modules and representation theory of groups would be useful, as well some familiarity with category theory, commutative diagrams and homological algebra.

Learning outcomes: As well as exposure to important topics in modern abstract algebra, this is a good place to gain familiarity with the use of category theory in mathematics.

Who might be interested: Likely to be of interest to mathematicians working on topics such as representation theory of finite dimensional algebras and finite groups, knot theory, Lie theory, algebraic topology, algebraic geometry, non-commutative geometry.

Practicalities: There may be an opportunity for participants to give short talks on topics that particularly interest them; this can be discussed during the course.

Outline

Prerequisites: As background, basic knowledge of rings, modules and representation theory of groups would be useful, as well some familiarity with category theory, commutative diagrams and homological algebra.

Learning outcomes: As well as exposure to important topics in modern abstract algebra, this is a good place to gain familiarity with the use of category theory in mathematics.

Who might be interested: Likely to be of interest to mathematicians working on topics such as representation theory of finite dimensional algebras and finite groups, knot theory, Lie theory, algebraic topology, algebraic geometry, non-commutative geometry.

Practicalities: There may be an opportunity for participants to give short talks on topics that particularly interest them; this can be discussed during the course.

Outline

Prerequisites: As background, basic knowledge of rings, modules and representation theory of groups would be useful, as well some familiarity with category theory, commutative diagrams and homological algebra.

Learning outcomes: As well as exposure to important topics in modern abstract algebra, this is a good place to gain familiarity with the use of category theory in mathematics.

Who might be interested: Likely to be of interest to mathematicians working on topics such as representation theory of finite dimensional algebras and finite groups, knot theory, Lie theory, algebraic topology, algebraic geometry, non-commutative geometry.

Practicalities: There may be an opportunity for participants to give short talks on topics that particularly interest them; this can be discussed during the course.

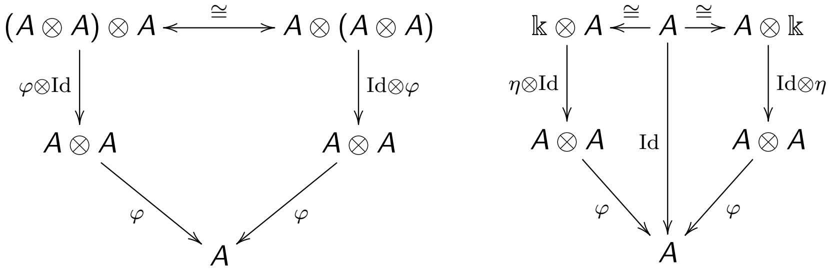
Quick introduction

A $\Bbbk$-*algebra* (A, φ, η) over a field $\Bbbk$ is a *monoid* in the monoidal category $(\mathbf{Vect}_{\Bbbk}, \otimes)$, i.e., a $\Bbbk$-vector space A with a *product* $\varphi\colon A \otimes A \to A$ and *unit* $\eta\colon \Bbbk \longrightarrow A$, which make the following diagrams in $\mathbf{Vect}_{\Bbbk}$ commute.

A is *commutative* if in addition the following diagram commutes.

Quick introduction

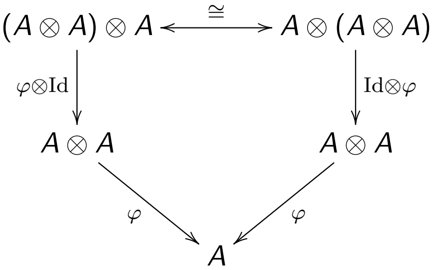
A $\Bbbk$-*algebra* (A, φ, η) over a field $\Bbbk$ is a *monoid* in the monoidal category $(\mathbf{Vect}_\Bbbk, \otimes)$, i.e., a $\Bbbk$-vector space A with a *product* $\varphi\colon A \otimes A \to A$ and *unit* $\eta\colon \Bbbk \longrightarrow A$, which make the following diagrams in $\mathbf{Vect}_\Bbbk$ commute.

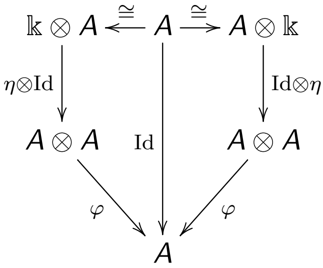


A is *commutative* if in addition the following diagram commutes.

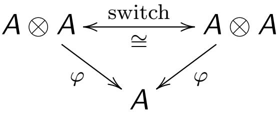
Quick introduction

A $\mathbb{k}$-*algebra* (A, φ, η) over a field $\mathbb{k}$ is a *monoid* in the monoidal category $(\mathbf{Vect}_{\mathbb{k}}, \otimes)$, i.e., a $\mathbb{k}$-vector space A with a *product* $\varphi\colon A \otimes A \to A$ and *unit* $\eta\colon \mathbb{k} \longrightarrow A$, which make the following diagrams in $\mathbf{Vect}_{\mathbb{k}}$ commute.





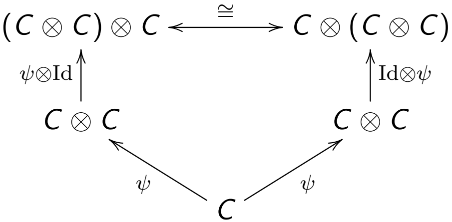
A is *commutative* if in addition the following diagram commutes.

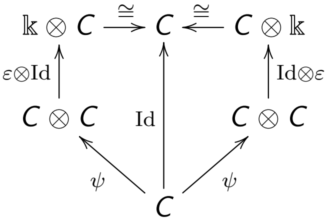


The dual notion is that of a $\mathbb{k}$-*coalgebra*, which is a triple (C, ψ, ε), with C a $\mathbb{k}$-vector space, a *coproduct* $\psi\colon C \to C \otimes C$, and a *counit* $\varepsilon\colon C \to \mathbb{k}$ fitting into the commutative diagrams shown.

This says that (C, ψ, ε) is a *comonoid* in $\mathbf{Vect}_{\mathbb{k}}$.
If the following diagram commutes then C is *cocommutative*.

The dual notion is that of a $\Bbbk$-*coalgebra*, which is a triple (C, ψ, ε), with C a $\Bbbk$-vector space, a *coproduct* $\psi\colon C \to C \otimes C$, and a *counit* $\varepsilon\colon C \to \Bbbk$ fitting into the commutative diagrams shown.

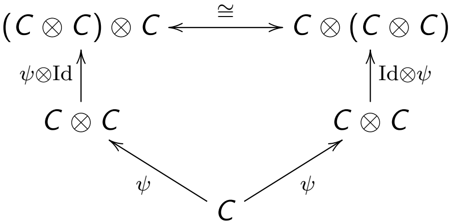


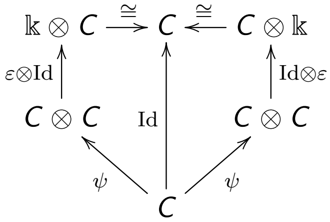


This says that (C, ψ, ε) is a *comonoid* in $\mathbf{Vect}_{\Bbbk}$.

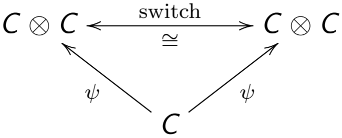
If the following diagram commutes then C is *cocommutative*.

The dual notion is that of a $\Bbbk$-*coalgebra*, which is a triple (C, ψ, ε), with C a $\Bbbk$-vector space, a *coproduct* $\psi\colon C \to C \otimes C$, and a *counit* $\varepsilon\colon C \to \Bbbk$ fitting into the commutative diagrams shown.





This says that (C, ψ, ε) is a *comonoid* in $\mathbf{Vect}_{\Bbbk}$.
If the following diagram commutes then C is *cocommutative*.



A $\Bbbk$-*Hopf algebra* H is a $\Bbbk$-vector space which is both an algebra and a coalgebra together with an antipode $\chi\colon H \to H$, so that all of this structure interacts in a certain way. A Hopf algebra is a *group object* in the category of coalgebras or a *cogroup object* in the category of algebras, so Hopf algebras generalise groups!

Some examples:

- The *group algebra* $\Bbbk[G]$ of a group G is a cocommutative Hopf algebra with the elements of G as a basis, product $\varphi(g_1 \otimes g_2) = g_1 g_2$, coproduct $\psi(g) = g \otimes g$ and antipode $\chi(g) = g^{-1}$.
- If G is a compact Lie group or more generally an H-space, $H_*(G;\Bbbk)$ and $H^*(G;\Bbbk)$ are Hopf algebras.
- If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.
- Quantum groups are Hopf algebras which are neither commutative nor cocommutative.
- Affine group schemes have associated commutative Hopf algebras.
- Examples from combinatorics.

A $\mathbb{k}$-*Hopf algebra* H is a $\mathbb{k}$-vector space which is both an algebra and a coalgebra together with an antipode $\chi\colon H \to H$, so that all of this structure interacts in a certain way. A Hopf algebra is a *group object* in the category of coalgebras or a *cogroup object* in the category of algebras, so Hopf algebras generalise groups!

Some examples:

- The *group algebra* $\mathbb{k}[G]$ of a group G is a cocommutative Hopf algebra with the elements of G as a basis, product $\varphi(g_1 \otimes g_2) = g_1 g_2$, coproduct $\psi(g) = g \otimes g$ and antipode $\chi(g) = g^{-1}$.
- If G is a compact Lie group or more generally an H-space, $H_*(G; \mathbb{k})$ and $H^*(G; \mathbb{k})$ are Hopf algebras.
- If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.
- Quantum groups are Hopf algebras which are neither commutative nor cocommutative.
- Affine group schemes have associated commutative Hopf algebras.
- Examples from combinatorics.

A $\Bbbk$-*Hopf algebra* H is a $\Bbbk$-vector space which is both an algebra and a coalgebra together with an antipode $\chi \colon H \to H$, so that all of this structure interacts in a certain way. A Hopf algebra is a *group object* in the category of coalgebras or a *cogroup object* in the category of algebras, so Hopf algebras generalise groups!

Some examples:

- The *group algebra* $\Bbbk[G]$ of a group G is a cocommutative Hopf algebra with the elements of G as a basis, product $\varphi(g_1 \otimes g_2) = g_1 g_2$, coproduct $\psi(g) = g \otimes g$ and antipode $\chi(g) = g^{-1}$.
- If G is a compact Lie group or more generally an H-space, $H_*(G;\Bbbk)$ and $H^*(G;\Bbbk)$ are Hopf algebras.
- If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.
- Quantum groups are Hopf algebras which are neither commutative nor cocommutative.
- Affine group schemes have associated commutative Hopf algebras.
- Examples from combinatorics.

A $\mathbb{k}$-*Hopf algebra* H is a $\mathbb{k}$-vector space which is both an algebra and a coalgebra together with an antipode $\chi\colon H \to H$, so that all of this structure interacts in a certain way. A Hopf algebra is a *group object* in the category of coalgebras or a *cogroup object* in the category of algebras, so Hopf algebras generalise groups!

Some examples:

- The *group algebra* $\mathbb{k}[G]$ of a group G is a cocommutative Hopf algebra with the elements of G as a basis, product $\varphi(g_1 \otimes g_2) = g_1 g_2$, coproduct $\psi(g) = g \otimes g$ and antipode $\chi(g) = g^{-1}$.
- If G is a compact Lie group or more generally an H-space, $H_*(G; \mathbb{k})$ and $H^*(G; \mathbb{k})$ are Hopf algebras.
- If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.
- Quantum groups are Hopf algebras which are neither commutative nor cocommutative.
- Affine group schemes have associated commutative Hopf algebras.
- Examples from combinatorics.

A $\Bbbk$-*Hopf algebra* H is a $\Bbbk$-vector space which is both an algebra and a coalgebra together with an antipode $\chi\colon H \to H$, so that all of this structure interacts in a certain way. A Hopf algebra is a *group object* in the category of coalgebras or a *cogroup object* in the category of algebras, so Hopf algebras generalise groups!

Some examples:

- The *group algebra* $\Bbbk[G]$ of a group G is a cocommutative Hopf algebra with the elements of G as a basis, product $\varphi(g_1 \otimes g_2) = g_1 g_2$, coproduct $\psi(g) = g \otimes g$ and antipode $\chi(g) = g^{-1}$.
- If G is a compact Lie group or more generally an H-space, $H_*(G;\Bbbk)$ and $H^*(G;\Bbbk)$ are Hopf algebras.
- If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.
- Quantum groups are Hopf algebras which are neither commutative nor cocommutative.
- Affine group schemes have associated commutative Hopf algebras.
- Examples from combinatorics.

A $\Bbbk$-*Hopf algebra* H is a $\Bbbk$-vector space which is both an algebra and a coalgebra together with an antipode $\chi\colon H \to H$, so that all of this structure interacts in a certain way. A Hopf algebra is a *group object* in the category of coalgebras or a *cogroup object* in the category of algebras, so Hopf algebras generalise groups!

Some examples:

- The *group algebra* $\Bbbk[G]$ of a group G is a cocommutative Hopf algebra with the elements of G as a basis, product $\varphi(g_1 \otimes g_2) = g_1 g_2$, coproduct $\psi(g) = g \otimes g$ and antipode $\chi(g) = g^{-1}$.
- If G is a compact Lie group or more generally an H-space, $H_*(G;\Bbbk)$ and $H^*(G;\Bbbk)$ are Hopf algebras.
- If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.
- Quantum groups are Hopf algebras which are neither commutative nor cocommutative.
- Affine group schemes have associated commutative Hopf algebras.
- Examples from combinatorics.

A $\Bbbk$-*Hopf algebra* H is a $\Bbbk$-vector space which is both an algebra and a coalgebra together with an antipode $\chi\colon H \to H$, so that all of this structure interacts in a certain way. A Hopf algebra is a *group object* in the category of coalgebras or a *cogroup object* in the category of algebras, so Hopf algebras generalise groups!

Some examples:

- The *group algebra* $\Bbbk[G]$ of a group G is a cocommutative Hopf algebra with the elements of G as a basis, product $\varphi(g_1 \otimes g_2) = g_1 g_2$, coproduct $\psi(g) = g \otimes g$ and antipode $\chi(g) = g^{-1}$.
- If G is a compact Lie group or more generally an H-space, $H_*(G;\Bbbk)$ and $H^*(G;\Bbbk)$ are Hopf algebras.
- If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.
- Quantum groups are Hopf algebras which are neither commutative nor cocommutative.
- Affine group schemes have associated commutative Hopf algebras.
- Examples from combinatorics.

A $\Bbbk$-*Hopf algebra* H is a $\Bbbk$-vector space which is both an algebra and a coalgebra together with an antipode $\chi\colon H \to H$, so that all of this structure interacts in a certain way. A Hopf algebra is a *group object* in the category of coalgebras or a *cogroup object* in the category of algebras, so Hopf algebras generalise groups!

Some examples:

- The *group algebra* $\Bbbk[G]$ of a group G is a cocommutative Hopf algebra with the elements of G as a basis, product $\varphi(g_1 \otimes g_2) = g_1 g_2$, coproduct $\psi(g) = g \otimes g$ and antipode $\chi(g) = g^{-1}$.
- If G is a compact Lie group or more generally an H-space, $H_*(G; \Bbbk)$ and $H^*(G; \Bbbk)$ are Hopf algebras.
- If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.
- Quantum groups are Hopf algebras which are neither commutative nor cocommutative.
- Affine group schemes have associated commutative Hopf algebras.
- Examples from combinatorics.

Outline of course content

Depending on time and the audience's interests, I expect to discuss most of the topics below, depending on the audience's interests.

- Some category theory: monoidal categories (vector spaces over a field as an important example), adjoint functors.
- Algebras and coalgebras over a field; bialgebras and Hopf algebras.
- Lots of examples.
- SubHopf algebras, adjoint actions and normal subHopf algebras.
- Modules and comodules, representation theory of a Hopf algebra.
- Hopf modules and finite dimensional Hopf algebras; every finite dimensional Hopf algebra is Frobenius.
- Quantum Groups.
- Homological algebra for modules over Hopf algebras.