SMSTC Geometry and Topology 2016-2017

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Aims

- Module 1
  - An overview of general topology
  - An introduction to algebraic topology
    (with an excursion into algebraic geometry)
- Module 2
  - An introduction to differential geometry:
    i.e. the geometry of manifolds
  - An introduction to differential topology:
    i.e. the topology of manifolds

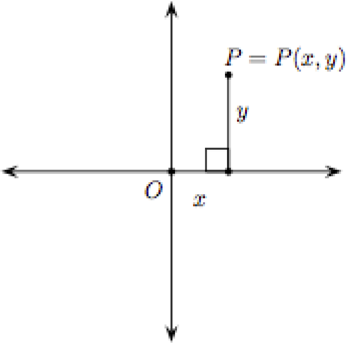
Prerequisites

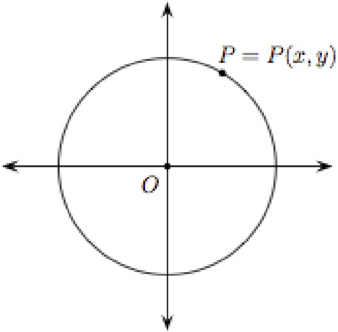
- Module 1
  - A course in metric spaces or topological spaces (or both). Important concepts: Open sets and neighbourhoods in metric spaces.
  - A course in group theory, including group actions, generators and relations, the structure theorem for finitely generated abelian groups.
- Module 2
  - Standard calculus courses. Some knowledge of vector calculus (e.g. div, grad, curl and Green's theorem) would be useful.
  - One or two basic courses in linear algebra. Important concepts: Abstract vector space, quotient vector spaces.

Topology = Topos + logus = "the logic of space"

- **Example:** the circle S^1 a fundamental 1-dimensional space.
- The circle sits inside $\mathbb{R}^2$, "carved out" by a distance equation.

$\mathbb{R}^2 =$ Cartesian plane $= \{(x, y)\}$ $\quad \mathbb{R}^2 \supset S^1 = \{(x, y) \,|\, x^2 + y^2 = 1\}$





Smooth manifolds

- **Definition:** An n-dimensional manifold is a topological space M which at each point $x \in M$ has an open neighbourhood $U \subseteq M$ homeomorphic to $\mathbb{R}^n$.
- **Example:** Euclidean space $\mathbb{R}^n$ is an n-dimensional manifold.
- An n-dimensional manifold is a space which can be obtained by glueing together copies of $\mathbb{R}^n$ using homeomorphisms (topological equivlaences).
- If the homeomorphisms are differentiable (and specified appropriately) then the manifold is smooth (or differentiable).
- Multivariable calculus extends to calculus on smooth manifolds.

Examples of manifolds 1

- Consider a differentiable function

$$f\colon \mathbb{R}^{n+k} \to \mathbb{R}^k.$$

With probability 1 the set

$$M := f^{-1}(x_1, \dots, x_k)$$

is a compact smooth n-manifold.

- **Example:** the unit n-sphere is the smooth n-manifold

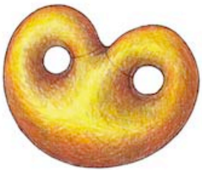
$$S^n := \left\{ (x_0, \dots, x_n) \;\middle|\; \sum_{i=0}^{n} x_i^2 = 1 \right\} \subset \mathbb{R}^{n+1}.$$

Take $f\colon \mathbb{R}^{n+1} \to \mathbb{R}, (x_0, \dots, x_n) \mapsto \sum_{i=0}^{n} x_i^2$.

Examples of manifolds 2: Surfaces

- A surface is a 2-dimensional compact manifold M.
- The genus g of M is the number of holes it has – not a proper mathematical definition as it stands!
- Here are the surfaces M_g with $g = 0, 1, 2$:





$M_0 = S^2 =$ sphere, $M_1 = S^1 \times S^1 =$ torus, $M_2 =$ pretzel.

Classification of manifolds

- Every connected compact 1-manifold is homeomorphic to S^1.
- The classification theorem for surfaces states that two connected surfaces are homeomorphic if and only if they have the same genus.
- Following the work of Thurston, Perelman and others, we have a comprehensive picture of 3-manifolds.
- The classification of 4-manifolds is wild.
- In dimensions $n \geq 4$, the classification of n-manifolds is "impossible" (but much can be said within prescribed classes of n-manifolds).

Probing spaces via maps from model spaces

- **Problem:** how to tell two spaces Y and Z apart?
- Idea: "probe" spaces by "counting" classes of continuous maps $f\colon X \to Y$, for model spaces X; e.g. $X = S^i$.
- **Definition:** Two continuous functions $f\colon X \to Y$ and $g\colon X \to Y$ are homotopic if there exists a family of continuous functions $h_t\colon X \to Y$ for $0 \leqslant t \leqslant 1$ such that

$$h_0 = f \ , \ h_1 = g \ : \ X \to Y$$

and $h_t(x)$ depends continuously on t and x.
Regard $\{h_t\}$ as a 'film' which starts at f and ends at g.

Homotopy groups: fishing for "spherical holes" in spaces

- Homotopy gives an equivalence relation on maps $f\colon X \to Y$.

$$[X, Y] := \{f\colon X \to Y\}/\text{homotopy}, \quad \pi_i(Y) := [S^i, Y].$$

- **Example:** $\pi_i(S^n) \cong \begin{cases} 0 & \text{if } i < n, \\ \mathbb{Z} & \text{if } i = n, \\ \text{finitely generated abelian} & i > n. \end{cases}$

- It follows that S^n is not homeomorphic to S^m, $n \neq m$.

Homology: fishing with a different net

- The homology groups of a topological space X are a sequence of abelian groups $H_i(X)$ for $i = 0, 1, 2, \ldots$.
- $H_i(X)$ measures the number of "i-dimensional holes" in X.
- Homology groups are harder to define but (frequently) easier to compute than homotopy groups.
- **Example 1**: $H_i(S^n) \cong \begin{cases} 0 & \text{if } i \neq 0 \text{ or } n, \\ \mathbb{Z} & \text{if } i = 0 \text{ or } n. \end{cases}$
- **Example 2**: $H_1(M_g) \cong \mathbb{Z}^{2g}$.
- It follows that M_g is not homeomorphic to M_h, $g \neq h$.

Geometry (Module 2) and Topology (Module 1)

- **Question:** how do the geometry and topology of spaces relate to one another?
- A smooth manifold M becomes a Riemannian manifold (M, g) through a choice of a Riemannian metric g on M.
- This structure makes it possible to assign a length to any smooth curve segment in M.
- When (M, g) is a Riemannian surface, we can define the curvature $K_g(x) \in \mathbb{R}$ for each $x \in M$.

Theorem: (Gauss-Bonnet) For (M, g) a Riemannian surface

$$\int_M K_g(x) dA = \chi(M) := \sum_{i=0}^{\infty} (-1)^i \mathrm{rank}(H_i(M)) = 2 - 2g.$$

Themes of the Geometry and Topology Stream

- Module 1
  - Topological spaces X, basic properties and invariants.
  - Homology groups $H_*(X)$ and some applications.
  - Surfaces: their role in algebraic geometry and their classification.

- Module 2
  - The differential geometry of curves and surfaces in $\mathbb{R}^3$.
  - The Gauss-Bonnet theorem, expressing $\chi(M)$ of a surface M as an integral of the curvature.
  - Vector calculus on smooth manifolds and deRham cohomology.

Some applications of Geometry and Topology

- “Space” is a powerful metaphor in many areas of mathematics and related fields. E.g.:
  - **Group theory:** Every group G defines a space BG, with $\pi_1(BG) = G$ and $\pi_i(BG) = 0$, $i > 1$.
  - **Algebraic geometry:** borrows techniques from differential geometry and algebraic topology to investigate algebraic varieties; solutions to algebraic equations.
  - **Theoretical physics:** gauge theory, general relativity, string theory, . . .
  - **Applied topology:** Topological concepts are important tools for finding pattern and structure in large data sets.

Relations with other SMSTC courses

- **Algebra:** Groups, both commutative and non-commutative, are ever-present in the Geometry/Topology course.
  Rings. If you are a commutative ring enthusiast, you might be pleased to know that many essential geometric constructions in our course can be reformulated in terms of commutative rings and their modules.
- **Pure Analysis:** Although we don't need Lebesgue integration theory, as developed in the Pure Analysis stream, integrals are of some importance in the Geometry/Topology course.
- **Applied Analysis and PDEs:** The third quarter of the Geometry/Topology course, on differential geometry, is somewhat related to the first quarter of the Applied Analysis course, on dynamical systems. An important class of dynamical systems (geodesic flows) comes from differential geometry.