Geometry and Topology 2017-2018

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Stream overview

(1) **Algebraic Topology**
- An overview of general topology by
  - Mark Grant from Aberdeen.
- An introduction to algebraic topology also by
  - Mark Grant.
- A short trip into algebraic geometry by
  - me.

(2) **Manifolds**
- An introduction to differential geometry by
  - Diletta Martinelli from Edinburgh,
  - Roberto Fringuelli also from Edinburgh.
- An introduction to differential topology by
  - Andrew Baker from Glasgow.

Prerequisites

(1) **Algebraic Topology**

- A course in metric spaces or topological spaces.
  - ★ Open and closed sets.
  - ★ Open neighbourhoods.
- A course in group theory.
  - ★ Group actions.
  - ★ Generators and relations.
  - ★ Finitely generated abelian groups.

(2) **Manifolds**

- Standard calculus courses.
  - ★ Basic vector calculus.
  - ★ Green's theorem.
- Basic courses in linear algebra.
  - ★ Abstract vector space.
  - ★ Quotient vector spaces.

Module 1: lectures

1. Basic examples and constructions of topological spaces.
2. Basic homotopy theory, homotopy groups and CW-complexes.
3. Cofibrations, cell attachments and CW-complexes.
4. Cellular approximation and relative homotopy groups.
5. Fibre bundles, fibrations and the Hopf map.
6. Algebraic curves and Riemann surfaces I.
7. Algebraic curves and Riemann surfaces II.
8. The definition of singular homology.
9. Singular homology: homotopy invariance and excision.
10. Computations and applications of singular homology.

Module 2: lectures

- Local manifolds and the implicit function theorem.
- Smooth curves and surfaces.
- Linear connections, Christoffel symbols and geodesics.
- The Theorema Egregium and curvature.
- The Gauss-Bonnet Theorem for surfaces.
- Differentiable manifolds, tangent bundle and vector fields.
- Differential forms, de Rham complex and Poincare Lemma.
- Differential forms, Stokes's Theorem and Poincare duality.
- The Levi-Civita connection and the Riemann curvature tensor.
- Thom isomorphism, intersection pairing, linking numbers.

Relations with other SMSTC courses

- Groups, Rings and Modules
  - ★ Fundamental groups are groups.
  - ★ Factoriality of rings via Poincare duality.
  - ★ Brauer group and cohomology.
- Algebras and Representation Theory
  - ★ Group actions gives linear representations in cohomology.
  - ★ Lefschetz fixed-point theorem.
  - ★ Noether problem via torsions in cohomology.
- Elliptic and Parabolic PDEs
  - ★ Hodge theory is proved using elliptic PDEs.
  - ★ Extremal metrics are solutions to PDEs.
  - ★ Atiyah–Singer index theorem.

Lüroth Problem

- Let $\mathbb{C}(x)$ be a field of rational function in 1 variable.
- Let $\mathbb{F}$ be a subfield in $\mathbb{C}(x)$ that contains $\mathbb{C}$.

Example

Let $\mathbb{F} = \mathbb{C}$.

Example

Let $\mathbb{F} = \mathbb{C}(x^2)$.

Example

Take any $f(x) \in \mathbb{C}(x)$ and let $\mathbb{F} = \mathbb{C}(f(x))$.

Question

Are there any other options for the subfield $\mathbb{F}$?

Theorem (Lüroth)

NO.

From fields to oriented surfaces

- The field $\mathbb{F}$ is generated by $f_1(x), \ldots, f_n(x)$ over $\mathbb{C}$.
- The functions $f_1(x), \ldots, f_n(x)$ are related by relations

$$\begin{cases} F_1\big(f_1, \ldots, f_n\big) = 0, \\ F_2\big(f_1, \ldots, f_n\big) = 0, \\ \ldots \\ F_r\big(f_1, \ldots, f_n\big) = 0. \end{cases}$$

- This gives a subset Σ in $\mathbb{C}^n$ given by

$$\begin{cases} F_1\big(x_1, \ldots, x_n\big) = 0, \\ F_2\big(x_1, \ldots, x_n\big) = 0, \\ \ldots \\ F_r\big(x_1, \ldots, x_n\big) = 0. \end{cases}$$

- One can choose generators of $\mathbb{F}$ such that Σ is *good*.

Classification of compact oriented surfaces

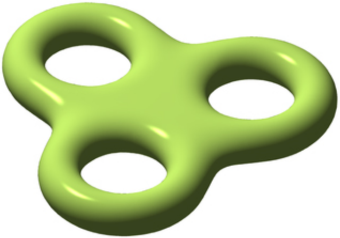
- The subset $\Sigma \subset \mathbb{C}^n$ is not compact.
- It can be *compactified* by squeezing $\mathbb{C}^n$ into $\mathbb{P}^n$.
- This gives a compact oriented surface S that contains Σ.
- Then $S \setminus \Sigma$ consists of finitely many points.

Theorem

The surface S is diffeomorphic to a sphere with g handles attached.

Example

If $g = 3$, then S looks like this:



Importance of being a sphere

Recall that $\mathbb{F}$ is a subfield in $\mathbb{C}(x)$ that contains $\mathbb{C}$.

Lemma

$\mathbb{F} = \mathbb{C}(f(x))$ *for some* $f(x) \in \mathbb{C}(x) \iff g = 0$.

Proof.

- If $\mathbb{F} = \mathbb{C}(f(x))$, then we obtain a map

$$\mathbb{C}^1 \to S$$

  which is one-to-one and *almost* surjective. This gives $g = 0$.
- If $g = 0$, then we must use Riemann–Roch theorem.

□

Since $\mathbb{F}$ is contained in $\mathbb{C}(x)$, we obtain a map

$$\mathbb{C}^1 \to S$$

which is *almost* surjective. If it is one-to-one, then we are done.

Euler characteristic

- We have a map $\mathbb{C}^1 \to S$ which is *almost* surjective.
- It gives a map $\phi\colon S^2 \to S$ that is surjective. This map is *good*.
- This means that away from finite subset $\Delta \subset S$ the subset

$$\phi^{-1}(P) \subset S^2$$

  consists of d points for every $P \in S \setminus \Delta$.
- If $d = 1$, then ϕ is bijection and we are done.

Triangulate the surface S such that Δ consists of vertices.

- Denote by $\mathbf{f}$ the number of faces.
- Denote by $\mathbf{e}$ the number of edges.
- Denote by $\mathbf{v}$ the number of vertices.

Then $\mathbf{v} - \mathbf{e} + \mathbf{f} = 2 - 2g$.

- Lift the triangulation to S^2 using ϕ.
- We have $d\mathbf{f}$ faces, $d\mathbf{e}$ edges and $\mathbf{v}' < d\mathbf{v}$ vertices.

Then $\mathbf{v}' - d\mathbf{e} + d\mathbf{f} = 2$. This gives $g = 0$, since

$$2 - 2g = \mathbf{v} - \mathbf{e} + \mathbf{f} = \frac{1}{d}\Big(d\mathbf{v} - d\mathbf{e} + d\mathbf{f}\Big) \geqslant \frac{1}{d}\Big(\mathbf{v}' - d\mathbf{e} + d\mathbf{f}\Big) = \frac{2}{d}.$$