SMSTC Stream : Applied Analysis and PDEs

Stream Leader: Lehel Banjai
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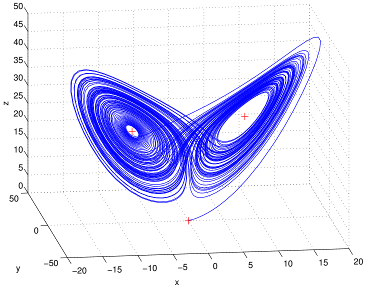
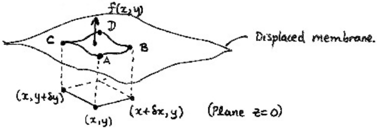


Lorenz ODEs



Displacement of a membrane

Applied Analysis and PDEs: Summary

- Provide a solid foundation to understand linear and nonlinear phenomena
- Give the language and tools to study the geometrical and qualitative properties of solutions to ODEs and PDEs
- The course assumes knowledge of undergraduate level ordinary differential equations, single- and multivariable real analysis, and linear algebra.

Module 1: Dynamical systems and conservation laws

  - Lectures 1-5 : ODEs
  - Lectures 6 : Overview of PDEs
  - Lectures 7-10 : Conservation Laws and Hyperbolic PDEs

Module 2: Elliptic and Parabolic PDE

  - Lectures 1-5: Parabolic PDEs
  - Lectures 6-10: Elliptic PDEs

- Two hand-in assessments per module.
- Contact: Lehel Banjai; `l.banjai@hw.ac.uk`

Content of the stream

Module 1: Dynamical Systems and Conservation Laws

- Geometry of dynamics and bifurcations of ODEs
- Stable and unstable manifolds, centre manifolds
- Scalar conservation laws, shock waves, uniqueness issues
- Systems of hyperbolic conservation laws

Example of a bifurcation

- The sound of a car brake.

A simple model of such a system

$$x' = \omega + x - x^3, \qquad \omega \text{ - (shifted) rotational speed.}$$

- For $-\frac{2}{3\sqrt{3}} < \omega < \frac{2}{3\sqrt{3}}$, three equilibria – two stable, one unstable.
- For $|\omega| > \frac{2}{3\sqrt{3}}$, single equilibrium.

ODE Example

- Lorenz Equations

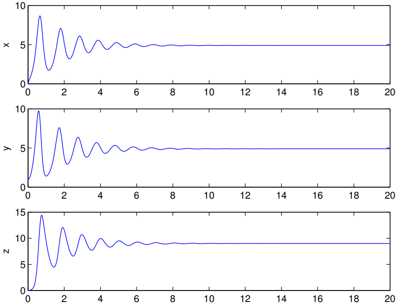
$$\left.\begin{array}{rcl} \frac{dx}{dt} & = & \sigma(y-x) \\ \frac{dy}{dt} & = & rx - y - xz \\ \frac{dz}{dt} & = & xy - bz \end{array}\right\} \qquad (1)$$

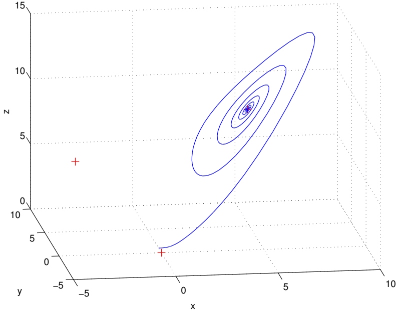
where σ, r, b are positive parameters.

- Simple looking system of ODEs derived by Lorenz to help analyse theoretical problems in meteorology and weather prediction.

Let's look at what happens to **same** initial data as r is increased.

Lorenz Equs: $r = 10$, $\sigma = 10, b = 8/3$



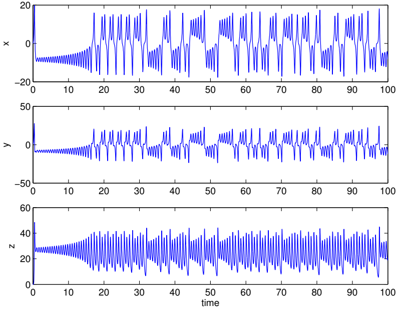


$x(t), y(t), z(t)$

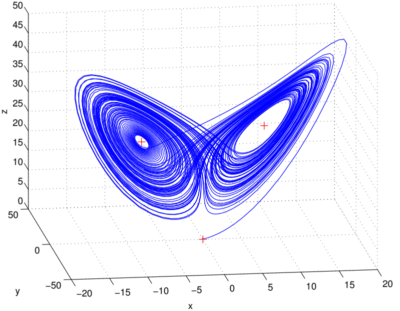
Phase space $= \mathbf{R}^3$

Solution converges to a 'fixed point'.

Lorenz Equs: $r = 28$, $\sigma = 10, b = 8/3$



$x(t), y(t), z(t)$



Phase space $= \mathbf{R}^3$

Solution converges to classic 'chaotic attractor'.

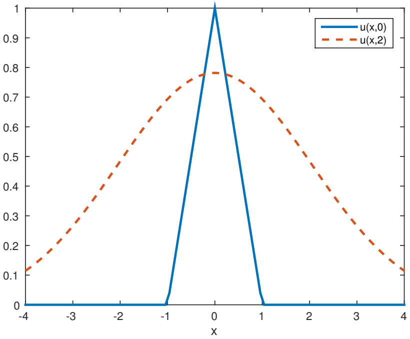
PDEs: Heat Equation

$$u_t = u_{xx}$$

This is a PDE: Infinite dimensional phase space, $u(\cdot, t)$.

PDEs: Heat Equation

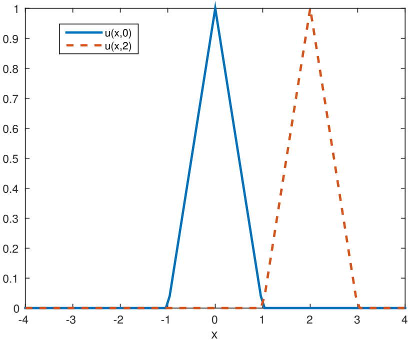
$$u_t = u_{xx}$$



PDEs: Wave Equation

Wave moving to the right at speed c:

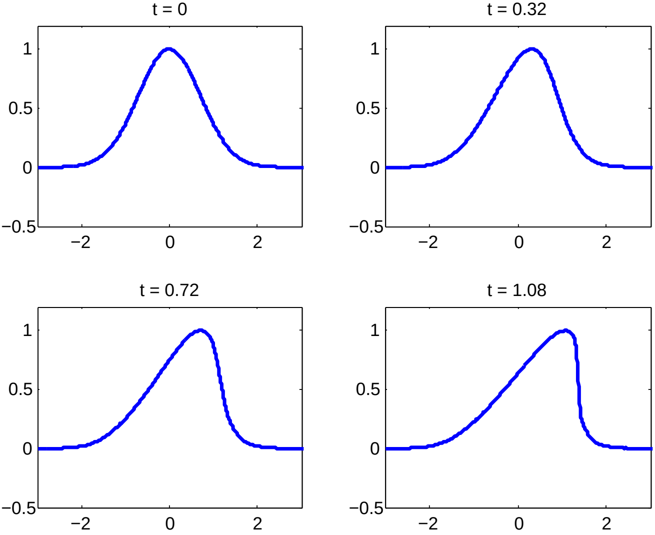
$$u_t + cu_x = 0$$



Burgers' equation: Shock development

Inviscid Burgers' equation

$$u_t + uu_x = 0, \qquad u(x,0) = \exp(-x^2).$$



Burgers' equation: Shock development

Adding diffusion Burgers' equation becomes

$$u_t + uu_x = \epsilon u_{xx}$$

Vanishing viscosity

- What happens when $\epsilon \to 0$?

Content of the stream

Module 2: Elliptic and Parabolic PDE

- Parabolic and Elliptic PDEs
- Travelling waves and similarity solutions
- Energy estimates and maximum principles
- Variational theory of PDEs
- Sobolev spaces

Poincaré constant

Poincaré inequality

Let $u \in C^2(\Omega)$, Ω a bounded domain and $u|_\Gamma = 0$, $\Gamma = \partial\Omega$. Then there exists $c > 0$ such that

$$\int_\Omega \|\nabla u\|^2 dx \geq c \int_\Omega |u|^2 dx.$$

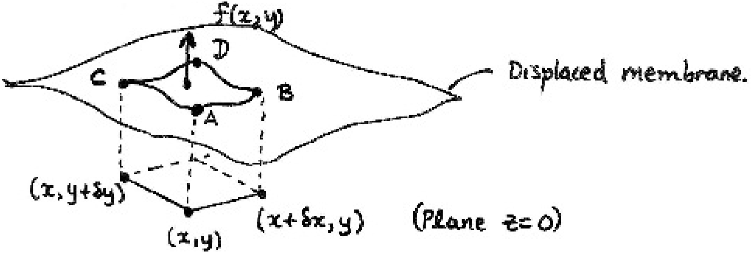
Poincaré constant

The optimal constant is $c = \lambda_1$, where λ_1 the least eigenvalue of

$$-\Delta u = \lambda u \text{ in } \Omega,$$
$$u = 0 \text{ on } \Gamma.$$

Displacement of a membrane

Consider a thin, elastic membrane attached to a wire frame. Model the displacement of the membrane under application of forces.



Membrane ctd.

- We describe the system as an optimization problem

$$\text{find } u \in V \; : \; J(u) \leq J(v) \quad \forall v \in V,$$

where

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega fu \, dx$$

and

$$V = \{v \in C^1(\overline{\Omega}) \; : \; v = g \text{ on } \partial\Omega\}.$$

- Next we reformulate it as a *variational equation*

$$\text{find } u \in V : \int_\Omega \nabla u \cdot \nabla w \, dx = \int_\Omega fw \, dx, \qquad \forall w \in V_0.$$

- Finally if u exists and $u \in C^2(\overline{\Omega})$ then

$$-\Delta u = f \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega.$$

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