SMSTC Stream : Applied Analysis and PDEs

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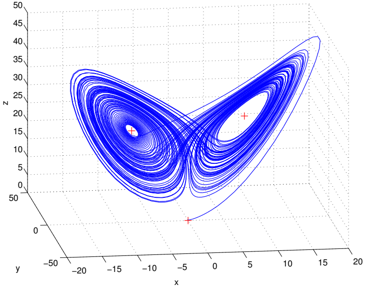
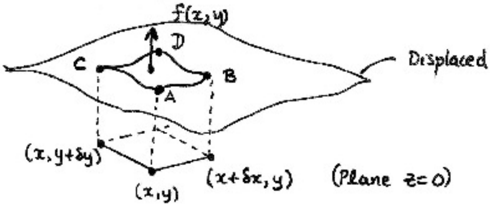


Lorenz ODEs



Displaced membrane

Applied Analysis and PDEs: Summary

Summary

- Provide a solid foundation and vocabulary to understand and discuss nonlinear phenomena.
- Tools to discuss the geometrical and qualitative properties of solutions to linear and nonlinear ODE and PDE.
- PDEs as models of physical processes, etc.

Prerequisites

- Undergraduate level ordinary differential equations, single- and multivariable real analysis, vector calculus, and linear algebra.

Module 1

- Lectures 1-5 : ODEs
- Lectures 6 : Overview of PDEs
- Lectures 7-10 : Conservation Laws and Hyperbolic PDEs

Module 2

- Lectures 1-5: Parabolic PDEs
- Lectures 6-10: Elliptic PDEs

- Two hand-in assessments per module.

Content of the stream

Module 1

- Geometry of dynamics and bifurcations of ODEs
- Stable and unstable manifolds, centre manifolds
- Scalar conservation laws, shock waves, uniqueness issues
- Systems of hyperbolic conservation laws

ODE Example

- Lorenz Equations

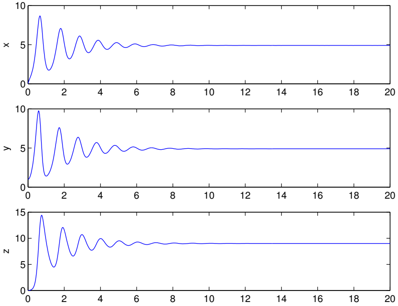
$$\left.\begin{array}{rcl} \frac{dx}{dt} & = & \sigma(y-x) \\ \frac{dy}{dt} & = & rx - y - xz \\ \frac{dz}{dt} & = & xy - bz \end{array}\right\} \qquad (1)$$

where σ, r, b are positive parameters.

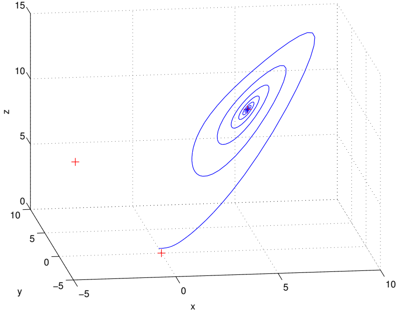
- Simple looking system of ODEs derived by Lorenz to help analyse theoretical problems in meteorology and weather prediction.

Let's look at what happens to **same** initial data as r is increased.

Lorenz Eqns: $r = 10$, $\sigma = 10, b = 8/3$



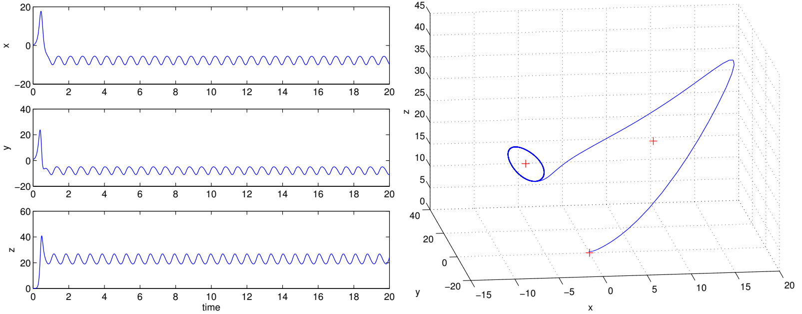
$x(t), y(t), z(t)$



Phase space $= \mathbb{R}^3$

Solution converges to a 'fixed point'.

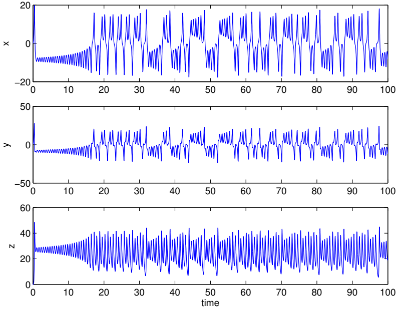
Lorenz Eqns: $r = 24.05$, $\sigma = 10, b = 8/3$

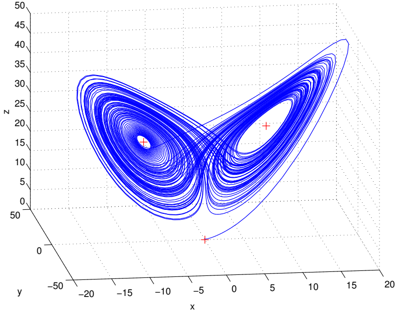


$x(t), y(t), z(t)$ Phase space $= \mathbb{R}^3$

Solution converges to a 'periodic orbit'.

Lorenz Eqns: $r = 28$, $\sigma = 10, b = 8/3$





$x(t), y(t), z(t)$

Phase space $= \mathbb{R}^3$

Solution converges to classic 'strange attractor'.

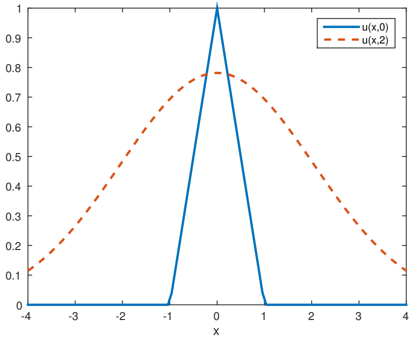
PDEs: Heat Equation

$$u_t(x,t) = u_{xx}(x,t)$$

This is a PDE: Infinite dimensional phase space, $u(\cdot, t)$.

PDEs: Heat Equation

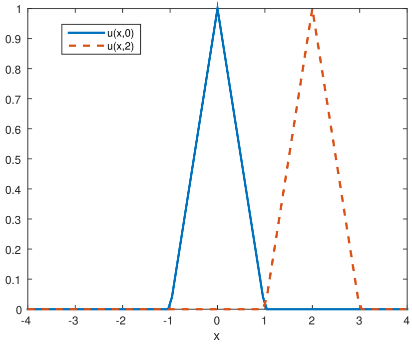
$$u_t(x,t) = u_{xx}(x,t)$$



PDEs: Wave Equation

Wave moving to the right at speed c:

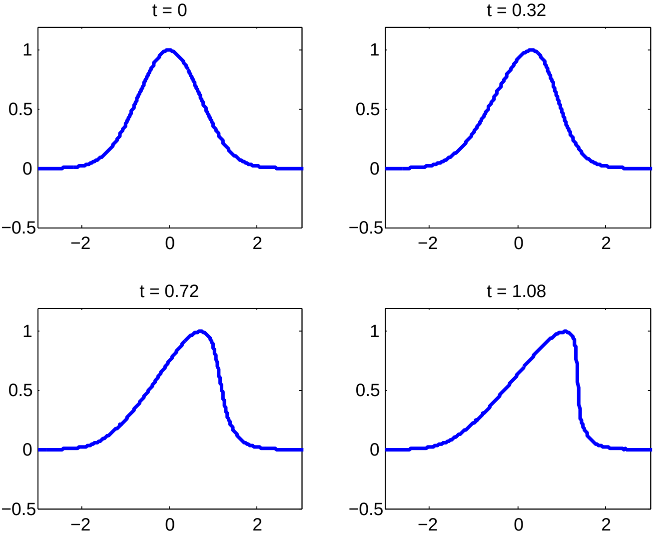
$$u_t(x,t) + c u_x(x,t) = 0$$



Burgers' equation: Shock development

Inviscid Burgers' equation

$$u_t + uu_x = 0, \qquad u(x,0) = \exp(-x^2).$$



Burgers' equation: Shock development

Adding diffusion Burgers' equation becomes

$$u_t + uu_x = \epsilon u_{xx}$$

Vanishing viscosity

- What happens when $\epsilon \to 0$?

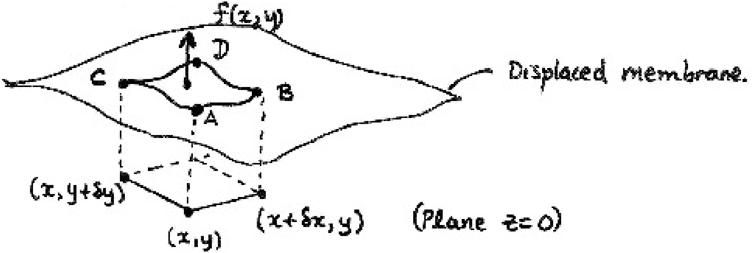
Content of the stream

Module 2

- Parabolic and Elliptic PDEs
- Travelling waves and similarity solutions
- Energy estimates and maximum principles
- Variational theory of PDEs
- Sobolev spaces

Displacement of a membrane

Consider a thin, elastic membrane attached to a wire frame. Model the displacement of the membrane under application of forces.



Membrane ctd.

- We describe the system as an optimization problem

$$\text{find } u \in V \ : \ J(u) \leq J(v) \quad \forall v \in V,$$

  where

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega fu\, dx$$

  and

$$V = \{v \in C^1(\overline{\Omega}) \ : \ v = g \text{ on } \partial\Omega\}.$$

- Next we reformulate it as a *variational equation*

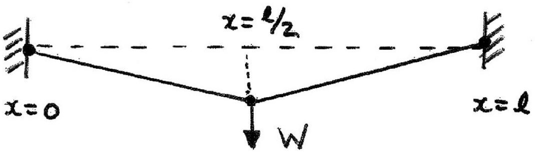
$$\text{find } u \in V : \int_\Omega \nabla u \cdot \nabla w\, dx = \int_\Omega fw\, dx, \qquad \forall w \in V_0.$$

- Finally if u exists and $u \in C^2(\overline{\Omega})$ then

$$-\Delta u = f \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega.$$

Which formulation to use?

To answer this question consider the loaded string:



- Variational equation

$$T \int_0^{\ell} u'(x)v'(x)dx = -Wv(\ell/2).$$

  - *The above picture satisfies this equation!*

- The PDE

$$-Tu''(x) = f(x), \qquad u(0) = u(\ell) = 0.$$

  - What is $f(x)$?
  - What is u''?

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- My e-mail: Lehel Banjai, `l.banjai@hw.ac.uk`