

Analysis theme: pure and applied

Heiko Gimperlein
h.gimperlein@hw.ac.uk

Heriot-Watt University

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2 pure modules:

- Measure and Integration (Semester I)
- Functional Analysis (Semester II)

2 applied modules:

- Dynamical Systems and Conservation Laws (Semester I)
- Elliptic and Parabolic PDEs (Semester II)

Integration as a tool

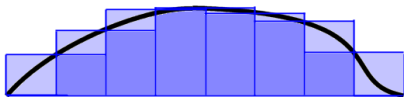
- Integration as a tool
 - Geometry, Representation Theory, Probability, Applied Analysis, etc...
- Geometry – Gauss Bonnet Theorem

$$\int_M K dA = 2\pi \chi(M)$$

- Applied analysis – solving differential equations
- Representation theory: average over compact group by integrating over the group.

Riemann integral:

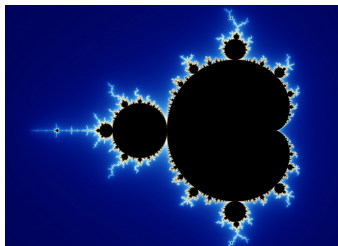
need to know measure of intervals / cubes $[a,b]^n$



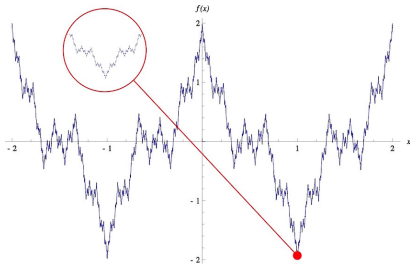
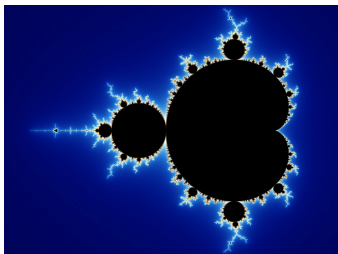
Lebesgue integral:

need to know measure of
sublevel sets $\{x : f(x) < a\}$

sublevel sets can be wild:



Sublevel sets can be wild
Can we define their volume?



There exist subsets of \mathbb{R}^n for which volume can't be defined
Banach-Tarski paradox decomposes ball into two balls,
using translations and rotations (using bizarre pieces)



→ Measure theory and integration

Abstract Measure and Integration

- Abstract measure spaces and Lebesgue integral
- Setting for Probability theory
 - A *measurable* set E corresponds to an **event**
 - A *measurable* function corresponds to a random variable
- Two lectures (foundations and complex measures)

This abstract framework will unify:

- Riemann and Lebesgue integral on \mathbb{R} : $\int_{\mathbb{R}} f(x) dx$
- Infinite series: $\sum_{n=1}^{\infty} a_n$
- Elements of $C(X)^*$ (the continuous dual space of $C(X)$):

$$\Lambda \in C(X)^* \implies \Lambda(f) = \int_X f(x) d\mu(x)$$

Constructing measures

- Construction of Lebesgue measure on real line
- Carathéodory construction.
 - using outer measures
 - fractal sets and Hausdorff dimensions
- Product measures: $X = Y \times Z$ (e.g. \mathbb{R}^2)
 - Fubini's theorem: when is it legitimate to change the order in a double integral.
- Radon measures – dual space of $C(X)$
- Total of 4 lectures (actually we'll construct Lebesgue measure on \mathbb{R} before developing the abstract framework).
- End with discussion of Fourier series and overview of where research goes from here.

2 pure modules:

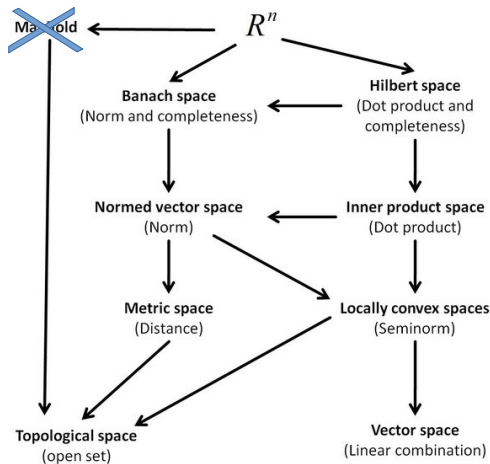
- Measure and Integration (Semester I)
- **Functional Analysis** (Semester II)

2 applied modules:

- Dynamical Systems and Conservation Laws (Semester I)
- Elliptic and Parabolic PDEs (Semester II)

Functional analysis:

Vector spaces and operators in infinite dimensions



Functional Analysis

Provides tools used right across mathematics: Language of analysis

- Banach and Hilbert spaces

$L^p(X)$ spaces, $C(X)$, $C(X)^*$, etc ...
Setting for Fourier series

- Linear operators and linear functionals
- Fundamental theorems: in the 'Scottish book' from cafe in Lwów.

Baire Category, Open Mapping, Uniform
Boundedness Principle, etc...

- Three lectures

Dual spaces and weak topologies

- Weak and weak* topologies
 - Banach-Alaoglu (weak* compactness of unit ball)
- Convexity in infinite dimensional setting.
 - Krein-Milman Theorem (very useful convexity result)
 - Use to produce Haar measure on compact groups.
- Two lectures

Spectral Theory

- Compact operators and their spectra
- General spectral theory

Banach algebras

Spectral theorem for self-adjoint operators

- Commutative C^* -algebras – Gelfand's theory
- Four lectures; again aiming to finish with overview lecture pointing towards current research in this direction.

Prerequisites for both pure courses

Formal prerequisites for both pure courses

- Undergraduate real analysis
 - Sequences, series, pointwise and uniform convergence.
 - Continuous and differentiable functions, basic properties.
- Metric space topology (at least in \mathbb{R}^d)
 - continuity of functions, open, closed and compact sets
- Countability
 - A set S is **countable** if $S = \{s_1, s_2, s_3, \dots\}$.
 - The set of reals \mathbb{R} is **uncountable**!

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For: Measure and Integration

- Will start with full account of Riemann integral, so self contained.
- If you've not seen the Riemann integral (or a version of the Lebesgue integral on \mathbb{R}) before, you'll need to be motivated.

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For: Functional Analysis

- If seen measure theory and some Fourier analysis before ideal!
- Without measure theory, should be able to get a lot out of the course, but need to ignore some of the examples.

Assessment for both courses

Ongoing feedback

- Exercises associated to each lecture.
- Not part of formal assessment; for discussion in tutorials.
- Some lectures (certainly including me) will ask for groups to present some of these in lectures.

Formal assessment for each course

- Two assignments for each course; normally consisting of three questions per assessment.
- In particular, there'll be no measure theory assumed in the functional analysis assessment

2 pure modules:

- Measure and Integration (Semester I)
- Functional Analysis (Semester II)

2 applied modules:

- **Dynamical Systems and Conservation Laws** (Semester I)
- Elliptic and Parabolic PDEs (Semester II)

Partial differential equations (PDE)

Study equations $\mathcal{F}(x, y, \dots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \dots, \frac{\partial^2 u}{\partial x \partial y}, \dots) = 0$

for a function $u = u(x, y, \dots)$, where \mathcal{F} is known and u is not.

Classical descriptions of natural phenomena:

- d'Alembert 1752, 1-dim. wave equation for vibrating string:

$$\partial_t^2 u(t, x) - \partial_x^2 u(t, x) = f(t, x)$$

- Euler 1759, D. Bernoulli 1762, wave equation for acoustics:

$$\partial_t^2 u(t, \vec{x}) - \Delta u(t, \vec{x}) = f(t, \vec{x}), \quad \Delta = \partial_{x_1}^2 + \dots + \partial_{x_n}^2$$

- Laplace 1780, Laplace equation for gravitational potential field

$$-\Delta u(\vec{x}) = f(\vec{x})$$

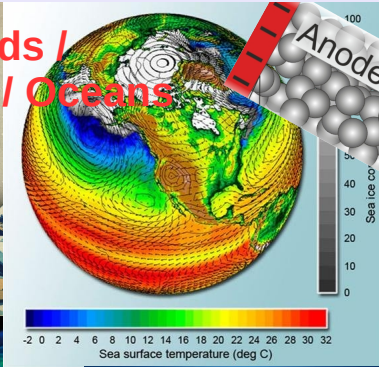
- Fourier 1822, heat equation

$$\partial_t u(t, \vec{x}) - \Delta u(t, \vec{x}) = f(t, \vec{x})$$

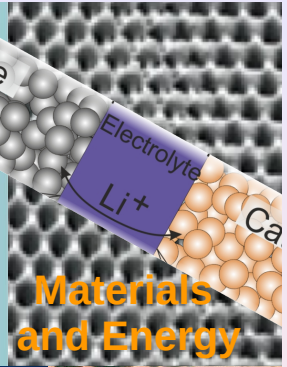
Some modern applications



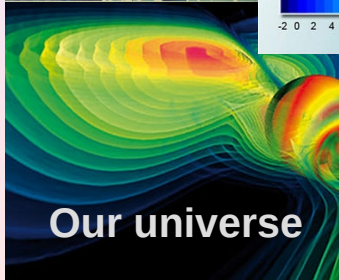
Fluids /
Weather / Oceans



-2 0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32
Sea surface temperature (deg C)



Materials
and Energy



Our universe

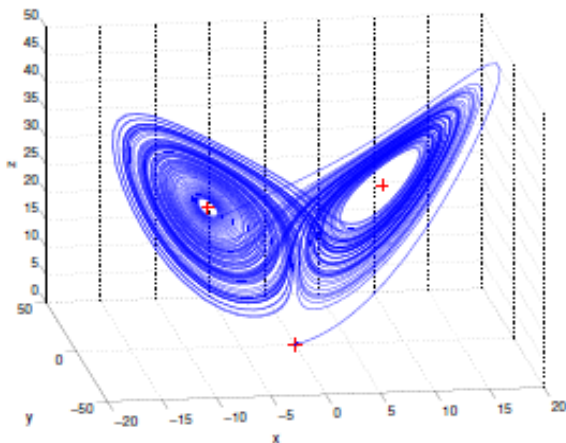


Biology / Medicine



Dynamical systems and conservation laws

Lorentz system of ODEs: weather forecasts



ODE Example

► Lorenz Equations

$$\left. \begin{aligned} \frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= rx - y - xz \\ \frac{dz}{dt} &= xy - bz \end{aligned} \right\} \quad (1)$$

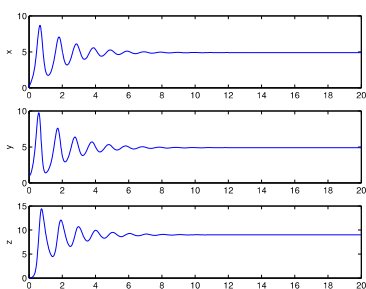
where σ, r, b are positive parameters.

► Simple looking system of ODEs derived by Lorenz to help analyse theoretical problems in meteorology and weather prediction.

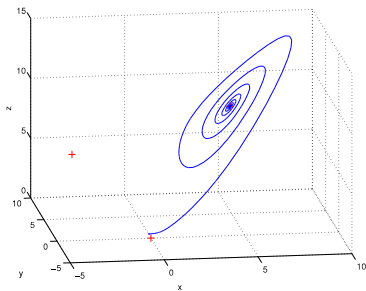
Based on a simplified model of convection: when a fluid is heated from below.

Let's look at what happens to **same** initial data as r is increased.

Lorenz Eqs: $r = 10, \sigma = 10, b = 8/3$



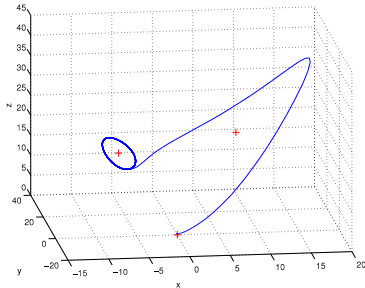
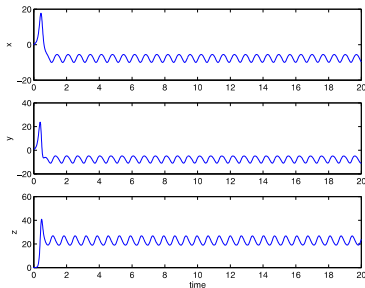
$x(t), y(t), z(t)$



Phase space = \mathbf{R}^3

Solution converges to a 'fixed point'.

Lorenz Eqs: $r = 24.05$, $\sigma = 10$, $b = 8/3$

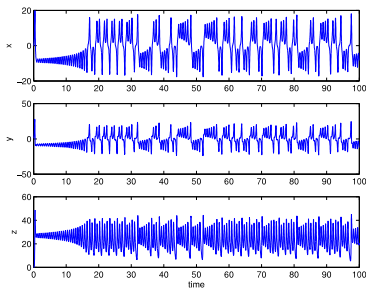


$x(t), y(t), z(t)$

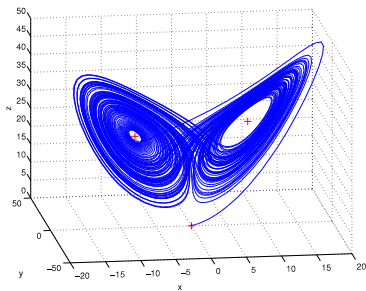
Phase space = \mathbf{R}^3

Solution converges to a 'periodic orbit'.

Lorenz Eqs: $r = 28, \sigma = 10, b = 8/3$



$x(t), y(t), z(t)$



Phase space = \mathbf{R}^3

Solution converges to classic 'chaotic attractor'.

Summary: Dynamical systems and conservation laws

- Dynamical systems and bifurcation
- Scalar conservation laws
- Systems of hyperbolic PDEs and shock waves

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Some basic questions

- **Modelling:**

What are the equations and where do they come from?

choose other modules

- **\exists solution?** Uniqueness? Dependence on data? *existence of classical solutions for Laplace and heat equations (applied treatment)*

Sobolev spaces and generalized weak solutions

- **Qualitative properties?**

self-similar solutions and travelling waves, maximum principles, Gidas-Ni-Nirenberg theorem on symmetry of solutions

- **numerical approximation** *mathematical background behind finite elements, for numerical methods choose other modules*

Formal prerequisites for both applied courses

- undergraduate ordinary differential equations
- single- and multivariable real analysis, linear algebra
- Semester II more or less self-contained, but Semester I useful for motivation and background.

Assessment for both applied courses

Ongoing feedback

- Exercises associated to each lecture.
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Formal assessment for each course

- Two (Sem I) / three (Sem II) assignment sheets, around 4 questions per assessment.

Conclusion

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Interactions: Pure maths is being applied, applied math solves pure problems.